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ON PARADOXES AND SITUATIONAL CONTEXT ANALYSIS[1]

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More and more often one comes across the view that the real source of many interpretational difficulties and obscurities is connected with paying too much attention to sentences, and at the same time neglecting the utterances, the convictions and other objects of this kind, as well as not taking into account the situational contexts of the examined utterances. Such a traditional approach leads to, among others, the antinomy of liar and many other paradoxes.

In a popular book, *Goodbye Descartes,* Keith Devlin (1998: 257) wrote:

> Once you take proper account of the context in which the Liar sentence is uttered, there is no more a paradox than there is a genuine conflict between the American who thinks that June is a summer month and the Australian who thinks June is a winter month. Here, laid bare, is what the Liar Paradox really amounts to.

This opinion after all, although characterised by gross exaggeration, can be considered as showing a certain general direction of the analysis of known paradoxes. In the article we present a discussion of two selected paradoxes: the ancient liar paradox and the contemporary Fitch's paradox. The approach presented herein will thereby take into account the situational contexts of the

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analysed utterances. As it will be demonstrated, both discussed paradoxes result from not taking into account the situational contexts of the analysed utterances, as well as from not discriminating between different situational contexts material for one and the same paradoxical utterance. It will also turn out that a useful turn tool for the performed analysis is a sort of multi-modal logic. We will call it situational modal logic and we will describe it in the last part of the article.

THE LIAR PARADOX

The liar paradox in its simplest form arises together with an utterance (conviction) that everything that it conveys is that and only that it itself is untrue. A reasoning that reveals the paradox is a consequence of two closely intertwined views.

Firstly, that each object with respect whereto we say that it is true or false, such as an utterance, a conviction, thought, sign or even a gesture, is an object which says something, states something or expresses something. Therefore there exists a sentence expressing its entire propositional content, i.e. in other words, everything and only that what this object expresses.

Secondly, the expression ”*is true*” is a predicate expression truth in its logical sense. Therefore, finding any expression or conviction to be true is equivalent to the acknowledging of everything that this utterance or conviction expresses. This last view we shall call *a rule of substantive correctness.*

Let letter ”T” replace the predicate ”*is true*” and ”$\sim$,” ”$\wedge$,” ”$\rightarrow$ ” and ”$\equiv$ ” will respectively be the symbols of negation, conjunction, and material implication and equivalence. The rule of substantive correctness may be written down as follows:

(T_0) For any freely determined utterance p and any freely determined sentence A:

$$(p \textit{ says that } A) \rightarrow \ (T(p) \equiv \ A)$$

The symbol p is here an individual (or specified description) and means an utterance, sign, thought, conviction and even a gesture or any other object, which is about something, expresses something or states something, i.e. of which it is possible to sensibly say whether it is true or false. Such an object will be generally referred to as an *utterance.* On the other hand, letter A represents a sentence expressing full propositional content of utterance p.

A reasoning leading to the Liar Paradox can be therefore presented as follows:

> There is such utterance p, that p *expresses that* $\sim T(p)$. Thanks to the rule of substantive correctness, if p *expresses that* $\sim T(p)$, then $(T(p) \equiv \ \sim T(p))$. Therefore, finally $T(p) \equiv \ \sim T(p)$.

Such an approach does not take into account the situational contexts of the examined utterance. It describes nether the situation in which the examined utterance p expresses that it is so and so, nor the situation of which the examined utterance p expresses that in this situation it is so and so.

Utterance in the form p *expresses that* A may be understood as an abbreviation of a more complex utterance. Namely: IN SITUATION S_K UTTERANCE P EXPRESSES THAT IN SITUATION S_L IT IS SO THAT A. Or better: IN SITUATION S_K IT IS SO THAT P EXPRESSES THAT IN SITUATION S_L IT IS SO THAT A.

The fact that in a given situation s_k it is so that A, will be recorded as $[s_k]\ A$. For any freely determined situation s_k the symbol $[s_k]$ plays therefore the role of a modal operator of a specific kind. We will call such symbols *situational modal operators.*

In order to reconstruct the discussed understanding, taking thereby into consideration the situational contexts of the examined utterance, one needs to construct above all the paraphrase of its prerequisite expressing that there exists such utterance p, that p *expresses that* $\sim T(p)$. Let us consider the two following paraphrases:

(P1) There is such utterance p and there exist such situations $s_k$and s_l that $[s_k]$ $(p$ *expresses that* $[s_l] \sim T(p))$

(P2) There is such utterance p and there exist such situations s_k and s_l that $[s_k]$ $(p$ *expresses that* $\sim [s_l]\ T(p))$

Both of the above phrases are made different by two situational contexts: (a) a situation where the examined utterance expresses that it is so and so, marked as s_k; (b) a situation where the examined utterance expresses that in its context it is so and so, marked as s_l.

Now we need to formulate a situational version of the rule of substantive correctness. Let us therefore assume that if in a given situation s_k utterance p expresses A, then in this situation taking p to be true is equivalent to acknowledgment of what p expresses, i.e. A. We will formulate this as follows:

(T) For any freely determined situation s_k utterance of a freely determined utterance p and a freely determined utterance A.

$$[s_k]\ (p \textit{ expresses that} \rightarrow\ [s_k]\ (T(p) \equiv\ A).$$

On the basis of prerequisite P1 we may now claim that

(1.1) $[s_k]\ (T(p) \equiv\ [s_l] \sim T(p))$.

On the basis of prerequisite P2 we may further claim that

(2.1) $[s_k]\ (T(p) \equiv\ \sim [s_l]\ T(p))$.

In order to conduct this reasoning further, we may however decide, which logical rights govern the situational modal operator, or — in other words, decide, which multimodal logic will be a relevant tool for the analysis of situational contexts of the examined utterances.

Let us firstly assume that

(Z1) Each situational modal operator is subject to the laws, which are the equivalents of the propositions of each normal modal logic. In other words, the logical laws, which govern the modal situational operators, are:

— all tautologies of the classical propositional calculus,

— all sentences in the form $[s_i]\ (A \rightarrow\ B) \rightarrow\ ([s_i]\ A \rightarrow\ [s_i]\ B)$,

and moreover for any freely determined sentences A and B,

— if $A \rightarrow\ B$ is a law and A is a law then B also is a law,

— if A is a law than for any i also $[s_i]\ A$ is a law.

Thanks to this, on the basis of prerequisite P1, we may further claim that

(1.2) $[s_k]\ T(p) \equiv\ [s_k]\ [s_l] \sim T(p)$

And on the basis of prerequisite P2 we may claim that

(2.2) $[s_k]\ T(p) \equiv\ [s_k] \sim [s_l]\ T(p)$

Let us assume further that the logical laws which govern the situational modal operators are also all sentences in the following form:

(Z2) $[s_i] \sim A \rightarrow\ \sim [s_i]\ A$ and

(Z3) $[s_i]\ A \equiv\ [s_j]\ [s_i]\ A$ and $\sim [s_i]\ A \equiv\ [s_j] \sim [s_i]\ A$.

On the basis of prerequisite P1 we may finally claim that

(1.3) $[s_k]\ T(p) \equiv\ [s_l] \sim T(p)$,

(1.4)$[s_k]\ T(p) \rightarrow\ \sim [s_l]\ T(p)$.

If therefore there exists such utterance p and there exist such situations s_k and s_l, and $[s_k](p$ *expresses that* $[s_l] \sim T(p))$ then $[s_k]\ T(p) \equiv\ [s_l] \sim T(p)$ and $[s_k]\ T(p) \rightarrow\ \sim [s_l]\ T(p)$.

Further if there exists such utterance p and such situation s_m that $[s_m](p$ *expresses that* $[s_m] \sim T(p))$ then we also have:

(1.5) $[s_m]\ T(p) \equiv\ [s_m] \sim T(p)$ and

(1.6) $[s_m]\ T(p) \rightarrow\ \sim [s_m]\ T(p)$ and

(1.7) $\sim [s_m]\ T(p)$ and

(1.8) $\sim [s_m] \sim T(p)$.

Situational reconstruction of reasoning resulting in the liar paradox on the basis of prerequisite P1 is therefore a proof of the proposition claiming that in certain situational contexts certain utterances are neither true or false.

On the basis of prerequisite P2 it may be claimed that:

(2.3) $[s_k]\ T(p) \equiv\ \sim [s_l]\ T(p)$.

If therefore there exists such utterance p and there exist such situations s_k and s_l, and $[s_k](p$ *expresses that* $\sim [s_l]\ T(p))$ then $[s_k]T(p) \equiv\ \sim [s_l]T(p)$.

In particular there exists such utterance p and such situation s_n that $[s_n](p$ *expresses that* $\sim [s_n]\ T(p))$, then $[s_n]\ T(p) \equiv\ \sim [s_n]\ T(p)$.

The situational reconstruction of the reasoning resulting in the lair paradox on the basis of prerequisite P2 may therefore be considered to be the proof for the claim on non-existence of a certain kind of situational context.

In the discussions concerning the liar paradox one distinguishes two versions of paradoxical utterances. An ordinary liar's utterance claiming that it itself IS false or IS untrue and the reinforced liar's utterance claiming that it itself IS NOT true. The first of the two is an utterance which ascribes something and the second is an utterance which denies something. The known analyses of both of these versions show that although the conviction that the ordinary liar's utterance is neither true nor false, liquidates the contradiction, yet however the conviction that he reinforced the liar's utterance is neither true nor false does not remove the contradiction (cf. Martin 1984).

Reconstruction of the liar's utterance based on prerequisite P1 may be therefore considered to be an equivalent of the ordinary liar's utterance, and the reconstruction based on prerequisite P2 is the equivalent of a reinforced liar's utterance. The former means existence of such utterance p and such situation s_m that $[s_m](p$ *expresses that* $[s_m] \sim T(p))$ and then both $\sim [s_m]T(p)$ and $\sim [s_m] \sim T(p)$. The latter would mean the existence of such utterance p and such situation s_n, that $[s_n](p$ *expresses that* $\sim [s_n]T(p))$, and this under the pain of contradiction is not possible.

The reconstructions of the reasoning resulting in the liar's paradox presented above, which took into account the situational contexts of the examined utterances, were based on three assumptions concerning the logic of the situational modal operators.

Firstly, we have assumed (Z1) that each situation modal operator is subject to laws which are the equivalents of the propositions of each normal modal logic. This assumption does not seem to raise any greater doubts. Sentences expressing that in SITUATION S_I IT IS SO THAT A, and the complex sentences constructed therefrom with the help of logical connectors, are subject to the laws of classical propositional calculus. Each situational modal operator also fulfils the equivalent of the axiom of regularity. If in a DETERMINED SITUATION S_I IT IS SO THAT A IMPLIES B AND IN SITUATION S_I IT IS SO THAT A, than IN SITUATION S_I IT IS SO THAT B. Moreover, if any sentence A is logically true, than it is true in all circumstances, therefore in any situation it is that A. The set of situational modal operators logic propositions is therefore closed with respect to the role equivalent to the rule of necessitation. We will call such equivalent *the rule of situational validity.* In view of the rule of situational validity, all laws of logic are valid in every situation. In particular, the law of excluded middle is valid IN EVERY SITUATION, i.e. for any i and any A in situation s_i it is so that A or $\sim A$. This does not mean, however, that all laws of logic apply to EVERY SITUATION, for example the law of excluded middle. The formula stating that for any i and any A in situation s_i it is so that any A or in situation s_i it is so that $\sim A$, is no longer a thesis of the considered logic. We already demonstrated earlier that there exists such utterance p and such situation s_m that $[s_m](p$ *expresses that* $[s_m] \sim T(p))$, then $\sim [s_m]\ T(p)$ and $\sim [s_m] \sim T(p)$, and therefore, in certain situational contexts some utterances are neither true, nor false.

In view of the assumption above, each situational modal operator is separable with respect to the conjunction connective.

Secondly, we have assumed (Z2) that a thesis of the situational modal operators logic is each sentence in the following form $[s_i] \sim A \rightarrow\ \sim [s_i]\ A$. This assumption states that IT IS IMPOSSIBLE FOR ANY SITUATION THAT IT IS SO THAT A AND SO THAT IT IS NOT TRUE THAT A. We will therefore call it the situational non-contradiction axiom.

Thirdly, we have assumed (Z3) that the theses of the logic of situational modal operators are all sentences in the form $[s_i]\ A \equiv\ [s_j]\ [s_i]\ A$ and $\sim [s_i]\ A \equiv [s_j] \sim [s_i]\ A$. If therefore IN SITUATION S_I IT IS SO THAT A, then IN ANY FREELY DETERMINED SITUATION S_J IT IS SO THAT IN SITUATION S_I IT SO THAT A, and if IN SITUATION S_J IT IS SO THAT IN SITUATION S_I IT IS SO THAT A, THEN IN SITUATION S_IIT IS SO THAT A. Similarly, if it is not true that IN SITUATION S_I IT IS SO THAT A, than IN ANY FREELY DETERMINED SITUATION S_J IT IS NOT TRUE THAT IN SITUATON S_I IT IS SO THAT A, and if IN SITUATION S_J IT IS NOT TRUE THAT IN SITUATON S_I IT IS SO THAT A, then it is not true that IN SITUATON S_I IT IS SO THAT A. Sentences stating that in a certain situation it is so and so, and that it is not true that in a certain situation is so and so are neutral with respect to situational contexts. Situational contexts of the analysed utterances were therefore treated as absolute contexts. We have therefore assumed that the logic of situational modal operators is the logic of absolute situational contexts.

The logic of situational modal operators, which meets the three above assumptions to be referred as to *situational modal logic.*

FITCH'S PARADOX

A situational analysis of Fitch's paradox was presented by Sten Lindstrom (Lindstrom 1997). His approach is in fact close to the above analysis of the liar's paradox. It is based on differentiating between situational contexts, material for an apt interpretation of the examined utterance.

Fitch's paradox (Fitch 1963) is an argument in favour of the thesis that IF THERE IS SUCH TRUE JUDGEMENT OF WHICH NO-ONE KNOWS THAT IT IS TRUE, THEN THERE ALSO IS SUCH TRUE JUDGEMENT, OF WHICH NO-ONE CAN SAY THAT IT IS TRUE. Since, undoubtedly, there are such judgements with respect to which it is unknown that they are true, one needs to reject the *cognizability principle*, according hereto every true judgement is cognizable.

Fitch's reasoning is as follows: Let A be such a true sentence, of which it is not known that it is true. Further, let B be the following sentence: A and it is not known that A. Sentence B is obviously true. What is more, there is no such situation in which it would be known that B. Let us assume that there is such situation s in which it is known that (A and it is not known that A). Since the epistemic operator *it is known that* is separable with respect to conjunction, in situation s (it is known that A and it is known that it is not known that A). Since for any freely determined A, IF IT IS KNOWN THAT A, THEN A, in situation s (it is known that A and it is not known that A). Therefore, there cannot exist such a situation in which it is known that B. Sentence B is therefore uncognizible.

According to Lindstrom, Fitch's understanding is based on equivocation, since it does not distinguish between the two following different situational contexts: (a) THE SITUATION, IN WHICH IT IS KNOWN THAT IT IS SO AND SO and (b) THE SITUATION, OF WHICH IT IS KNOWN THAT IT IS SO AND SO. If one only observes this distinction, then in Lindstrom's opinion the utterance stating that IN A CERTAIN SITUATION IT IS KNOWN THAT IN A CERTAIN (OTHER) SITUATION IT IS SO AND SO AND THAT IN THIS EXACT SITUATION THIS IS NOT KNOWN, ceases to be paradoxical. In order to make this distinction more apparent, Lindstrom provides the following example.

Today John knows that yesterday there was an even number of books in his book cabinet and that then he did not know that.

Let us analyse this example with the use of situational modal logic used earlier for the analysis of the liar's paradox. Let us assume that A means the sentence *There is an even number of book's in John's book cabinet.* Let us further assume that $s_d$means the situation today and s_w means the situation yesterday. Moreover, K_J will mean *John knows that.* The discussed sentence may be then written down as follows:

$[s_d]\ K_J\ ([s_w]\ A \wedge [s_w] \sim K_J\ [s_w]\ A),$

or, if only the epistemic operator K_J is subject to the extensionality rule, in the following form, equivalent on the basis of the situational modal logic:

$[s_d]\ K_J\ [s_w]\ (A \wedge [s_w] \sim K_J\ [s_w]\ A).$

Let K mean the epistemic operator *it is known that.* Generally, the fact that in situation $s_i$it is known that A, will be written down as $[s_i]\ KA$, the fact that it is known that in situation $s_j$it is so that it is known that A will be written down as $K\ [s_j]\ A$, and the fact that in situation $s_i$it is known that in situation $s_j$it is so that A will be written down as $[s_i]\ K\ [s_j]\ A$.

It may now be demonstrated that the existence of such sentence A and such situation s, that IN SITUATION S IT IS SO THAT A, AND THAT IN SITUATION S IT IS NOT KNOWN THAT IN SITUATION S IT IS SO THAT A, is not at all contradictory with the cognizability principle. We need to, however, formulate a situational paraphrase of the cognizability principle stating that each true judgement is cognizable. Let us namely assume that IF IN ANY FREELY DETERMINED SITUATION S_I IT IS SO THAT A, THAN THERE IS SUCH SITUATION S_J IN WHICH IT IS KNOWN THAT IN SITUATION S_I IT IS SO THAT A. In other words, let us assume that:

(K) For any i, if $[s_i]\ A$, then there exists such j that $[s_j]\ K\ [s_i]\ A$.

Let us also assume that the knowledge operator K is separable with respect to conjunction and that the knowledge logically implies the truth, i.e. that operator K is governed by the following laws:

(K1) $K\ (A \wedge B) \equiv\ K(A) \wedge K(B)$,

(K2) $K(A) \rightarrow\ A$.

Now, let us assume that s_k is such a situation, and A is such a sentence that:

(1.1) $[s_k]\ (A \wedge \sim K\ [s_k]\ A)$.

Thanks to the situational version of the cognizability principle, we may now claim that for certain determined l

(1.2) $[s_l]\ K\ [s_k]\ (A \wedge \sim K\ [s_k]\ A)$.

Thanks to the assumptions concerning the logic of the situational model operators (Z1), (Z2) and (Z3) and the knowledge operator (K1) and (K2) we may in turn claim that

(1.3) $[s_l]\ K\ [s_k]\ A \wedge [s_k] \sim K\ [s_k]\ A$

and that

(1.4) $[s_l]\ K\ [s_k]\ A \wedge \sim [s_k]\ K\ [s_k]\ A$.

This is no contradiction, of course. Simply, in situation $s_l$it is so that A, and in situation $s_k$it is not known that in situation $s_k$it is so that A.

It may be demonstrated, however that there does not exist such situation s_m in which it is known that in situation s_m it is so that A and in situation s_m it is not known that in situation s_m it is so that A. When applying the situational modal logic and laws K1 and K2 it is possible to prove any sentence in the form:

$\sim [s_m]\ K\ [s_m]\ (A \wedge \sim [s_m]\ KA)$.

Let us assume not directly that for a certain m

(2.1) $[s_m]\ K\ [s_m]\ (A \wedge \sim [s_m]\ KA)$

Thanks to (Z1), (Z2) and (Z3) and (K1) and (K2) we now have

(2.2) $[s_m]\ K\ [s_m]\ A \wedge [s_m] \sim K\ [s_m]\ A$

and

(2.3) $[s_m]\ K\ [s_m]\ A \wedge \sim [s_m]\ K\ [s_m]\ A$.

Taking into account the situational contexts with the use of the previously presented situational modal logic makes it therefore possible to demonstrate that Fitch's argument does not at all undermine the moderate version of the cognizability principle, according whereto IF IN A GIVEN SITUATION IT IS SO AND SO, THEN THERE IS ALSO SUCH ANOTHER SITUATION IN WHICH IT IS KNOWN THAT IN THE FIRST SITUATION IT IS SO AND SO.

SITUATIONAL MODAL LOGIC

The discussion presented above concerning the liar's paradox and Fitch's paradox indicates that the situational modal logic used therein is an interesting tool for analysing situational contexts of the examined utterances.

Hereinafter, this logic will have the form of a formalized propositional calculus. First, we will present the symbolic language of this logic, and then its syntactic and semantic characteristic. We will also define a set of propositions of the situational modal logic. Then we will introduce the notions of the situational model and the situational modal tautology. It will finally turn out that each correctly constructed expression of the language of situational modal logic is a proposition if and only if it is a situational modal tautology. The syntactic and semantic approaches therefore characterise the same set of logical theses.

The language of situational modal logic (**SLL**) is obtained by enrichment of the dictionary of the classic propositional calculus by countably many one-argument modal operators: $[s_0]$, $[s_1]$, $[s_2]\dots$ It therefore contains only the following symbols:

(S1) countably many sentence symbols: P_0, P_1, $P_2\dots$,

(S2) the connectives of the classic propositional calculus $\sim$, $\wedge$, $\rightarrow$,

(S3) countably many one-argument modal operators: $[s_0]$, $[s_1]$, $[s_2]\dots$, (S4) brackets: (,).

A set of **SLL** well-formed formulas, or formulas in short, is defined inductively in the usual way. Letters A, B, C ... will mean freely determined correctly constructed formulas. Symbols $[s_i]$ and $[s_j]$ etc. will mean respectively the i-th and the *j-th* situational operator, and symbol $P_i$will mean the i-th sentence symbol.

The formula in the form of $[s_i]$ A should be read: *in situation s_i it is so that* A.

The formula in the form $\sim(\sim A \wedge \sim B)$ will also be written down as $A \vee B$, and the formula in the form $(A \rightarrow B) \wedge (B \rightarrow A)$ will be also written down as $A \equiv B$.

The set of propositions of the situational modal logic (**SLA**), or the propositions in short, is the smallest containing:

(A1) all of the tautologies of the classic propositional calculus,

(A2) all of the formulas in the following form $[s_i]\ (A \rightarrow B) \rightarrow ([s_i]\ A \rightarrow [s_i]\ B)$,

(A3) all of the formulas in the following form $[s_i]\ A \rightarrow \sim [s_i] \sim A$,

(A4) all of the formulas in the following form $[s_i]\ A \equiv [s_j]\ [s_i]\ A$ and $\sim [s_i]\ A \equiv [s_j] \sim [s_i]\ A$,

and closed on:

(R1) *modus ponens* $A \rightarrow B$, A / B,

(R2) the rule of situational validity A / $[s_i]$ A.

We will say that formulas A and B are equivalent, if formula $A \equiv B$ is a proposition.

Conclusion 1

(1) Each formula in the following form: $[s_i]\ (A_0 \wedge A_1 \wedge ... \wedge A_n) \equiv [s_i]\ A_0 \wedge [s_i]\ A_1 \wedge ... \wedge [s_i]\ A_n$ is a proposition.

(2) A set of propositions is closed for the extensionality rule $A \equiv B$ / $[s_i]\ A \equiv [s_i]\ B$ and the monotonicity rule $A \rightarrow B$ / $[s_i]\ A \rightarrow [s_i]\ B$.

Conclusion 2. Each formula in the following form: $F_0F_1...F_n\ [s_i]\ B$ in which $0 \leqslant k \leqslant n$ is a negation connective or a freely determined situational modal operator, is equivalent to the formula in the following form: $[s_i]$ B or $\sim [s_i]$ B.

Let A be a formula in the following form $[s_i]\ B$ or $\sim[s_i]\ B$. If formula A is preceded by the symbol of negation or a situational modal marker then on the basis of A4 we will get a formula equivalent to the formula in the form $[s_i]\ B$ or $\sim [s_i]\ B$.

Conclusion 3. If A is a proposition or a counterproposition then each equivalent in the form of $[s_i]\ A \equiv\ A$ is a proposition.

If A is a proposition, then on the basis of the situational applicability rule also $[s_i]\ A$ is a proposition. If A is a counterproposition than $\sim A$ is a proposition, on the basis of the situational applicability rule $[s_i] \sim A$ is a proposition and thanks to A3 $\sim [s_i]\ A$ is a proposition.

Conclusion 4.

(1) If A is a formula in the following form $[s_j]\ B$, then on the basis of A4 each equivalent in the form of $[s_i]\ A \equiv\ A$ is a proposition.

(2) If each equivalent in the form of $[s_i]\ A \equiv\ A$ is a proposition and each equivalent in the form of $[s_i]\ B \equiv\ B$ is a proposition, then each equivalent in the form of $[s_i] \sim A \equiv\ \sim A$, $[s_i]\ (A \wedge B) \equiv\ A \wedge B$ and $[s_i]\ (A \to\ B) \equiv\ A \to\ B$ is also a proposition.

Let us assume that each equivalent in the following form $[s_i]\ A \equiv\ A$ is a proposition and each equivalent in the following form $[s_i]\ B \equiv\ B$ is a proposition.

Therefore, each equivalent in the form of $\sim A \equiv\ \sim [s_i]\ A$ is a proposition, moreover each equivalent in the form of $[s_j] \sim A \equiv\ [s_j] \sim [s_i]\ A$ is a proposition and each equivalent in the form of $[s_j] \sim A \equiv\ \sim [s_i]\ A$ is a proposition. Therefore, each equivalent in the form of $[s_j] \sim A \equiv\ \sim A$ is a proposition.

Similarly, each proposition in the form of $(A \wedge B) \equiv\ ([s_i]\ A \wedge [s_i]\ B)$ is a proposition and therefore each equivalent in the form of $(A \wedge B) \equiv\ [s_i]\ (A \wedge B)$ is a proposition.

Finally, since each implication in the form of $[s_i]\ (A \to\ B) \to\ ([s_i]\ A \to [s_i]\ B)$ is a proposition, then each implication in the form of $[s_i]\ (A \to\ B) \to (A \to\ B)$ is a proposition. Since the implications in the form of $B \to\ (A \to B)$ and $\sim A \to\ (A \to\ B)$ are classical prepositional calculus sentences, then implications in the form of $[s_i]\ B \to\ [s_i]\ (A \to\ B)$ and $[s_i] \sim A \to\ [s_i]\ (A \to B)$ are propositions. Therefore, each implication in the form of $B \to\ [s_i]\ (A \to B)$ and each implication in the form of $\sim A \to\ [s_i]\ (A \to\ B)$ is a proposition, and therefore each implication in the form of $(A \to\ B) \to\ [s_i]\ (A \to\ B)$ is a proposition. Therefore, finally, each equivalent in the form of $[s_i]\ (A \to\ B) \equiv\ A \to\ B$ is a proposition.

Let us inductively define property N.

(0) Each proposition and counterproposition has property N.

(1) Each formula in the form of $[s_i]\ A$ has property N.

(2) If formulas A and B have property N, then formulas $\sim A$, $\sim B$, $A \wedge B$ and $A \rightarrow B$ also have property N.

(3) Nothing else has property N.

CONCLUSION 5. If formula A has property N, then each equivalent in the form of $[s_i]\ A \equiv A$ is a proposition.

The fact that in a certain situation $s_i$it is so that A, will be understood by us in such a manner that in any circumstances in which situation $s_i$takes place, sentence A is true. In other words, we assume that sentence $[s_i]\ A$ is true if and only if sentence A is true in every possible word, of which situation $s_i$is a part. This concept will be the starting point for the semantic description of the situational modal logic.

We will understand a "situational model" as an ordered triple $\langle W, \lambda, V \rangle$, in which W is a not empty set, λ is a sequence of not empty sub-sets of W, and V is a function ascribing each sentence symbol a certain sub-set of set W.

We will call the elements of set W possible worlds and we will mark them with the following symbols, v, w, etc. W_i^λshall mean the *i-th* element in sequence λ. We will call set W_i^λ a set of possible worlds in which situation $s_i$takes place. The assumption that for any freely determined i set W_i^λ is not empty, reflects the conviction that each situation takes place in a certain possible world. For any freely determined i we will call set $V(P_i)$ a set of possible worlds, in which sentence $P_i$is true.

Notation $w \models A$ shall mean that formula A is true in a possible world w.

Let $\langle W, \lambda, V \rangle$ be a determined situational model. For any freely determined world w belonging to W:

$w \models P$ iff $w \in V(P_i)$;

$w \models \sim A$ iff it is not true that $w \models A$;

$w \models A \wedge B$ iff $w \models A$ and $w \models B$;

$w \models A \rightarrow B$ iff $w \models A$ then $w \models B$;

$w \models [s_i]\ A$ iff $\forall v$ if $v \in W_i^\lambda$ then $v \models A$.

Let us say that formula A is valid in situational model $\langle W, \lambda, V \rangle$, if for any w belonging to W, $w \models A$. Let us also say that formula A is a situational modal tautology, if it is valid in every situational model.

CONCLUSION 6. All propositions of the modal situational logic are situational modal tautologies.

All tautologies of the classic propositional calculus and all formulas in the form: $[s_i]\ (A \rightarrow B) \rightarrow ([s_i]\ A \rightarrow [s_i]\ B)$, $[s_i]\ A \rightarrow \sim [s_i] \sim A$, $[s_i]\ A \equiv [s_j]\ [s_i]\ A$ and $\sim [s_i]\ A \equiv [s_j] \sim [s_i]\ A$ are applicable in every situational model. Moreover, if the formula in the form $A \rightarrow B$ is applicable in every situational model and formula A is applicable in every situational model, then also formula

B is applicable in every situational model. Similarly, if formula A is applicable in every situational model, then each formula in the form $[s_i]\ A$ also applies in every situational model.

In order to more easily see that all formulas in the form $[s_i]\ A \equiv\ [s_j]\ [s_i]\ A$ and $\sim [s_i]\ A \equiv\ [s_j] \sim [s_i]\ A$ are situational modal tautologies, let us notice that in any freely determined situational model any freely determined formula in the form $[s_i]\ A$ is true in a certain possible world, if and only if it is true in all possible worlds.

For a freely determined $\langle W, \lambda, V\rangle$ we therefore have:

(a) $\exists w\ (w \models [s_i]\ A)$ iff $\forall w\ (w \models [s_i]\ A)$.

For a freely determined $\langle W, \lambda, V\rangle$ we also have:

(b) if $\forall w\ (w \models A)$ then $\exists w\ (w \models A)$ iff $\forall w\ (w \models A)$,

(c) if $\sim \exists w\ (w \models A)$, then $\exists w\ (w \models A)$ iff $\forall w\ (w \models A)$,

(d) $\exists w\ (w \models A)$ iff $\forall w\ (w \models A)$ and $\exists w\ (w \models B)$ iff $\forall w\ (w \models B)$, then $\exists w\ (w \models \sim A)$ iff $\forall w\ (w \models \sim A)$, $\exists w\ (w \models A \wedge B)$ iff $\forall w\ (w \models A \wedge B)$ and $\exists w\ (w \models A \rightarrow\ B)$ iff $\forall w\ (w \models A \rightarrow\ B)$,

(e) $\exists w\ (w \models A)$ iff $\forall w\ (w \models A)$, iff for any freely determined i $\forall w\ (w \models [s_i]\ A \equiv\ A)$.

Points (a), (b) and (c) obtain on the basis of the definition of truth in the situational model and thanks to the non-emptiness of set W.

Let us now assume that $\exists w\ (w \models A)$ iff $\forall w\ (w \models A)$ and $\exists w\ (w \models B)$ iff $\forall w\ (w \models B)$.

Let $\exists w\ (w \models \sim A)$, therefore $\exists w$ (not true that $w \models A$), i.e. that it is not true that $\forall w\ (w \models A)$, and therefore it is not true that $\exists w\ (w \models A)$, and therefore finally $\forall w\ (w \models \sim A)$. Let further $\forall w\ (w \models \sim A)$, therefore not true that $\exists w\ (w \models A)$, and therefore not true that $\forall w\ (w \models A)$, and finally $\exists w\ (w \models \sim A)$.

Let $\exists w\ (w \models A \wedge B)$, therefore $\exists w\ (w \models A)$ and $\exists w\ (w \models B)$, and therefore $\forall w\ (w \models A)$ and $\forall w\ (w \models B)$, i.e. $\forall w\ (w \models A$ and $w \models B)$ and finally $\forall w\ (w \models A \wedge B)$. Further let $\forall w\ (w \models A \wedge B)$, and therefore also $\exists w\ (w \models A \wedge B)$.

Let us further assume that $\exists w\ (w \models A)$ iff $\forall w\ (w \models A)$. Let's now assume that $w_0 \models A$. Therefore $\exists w\ (w \models A)$, and also $\forall w\ (w \models A)$, and therefore for any freely determined i $\forall w$ (if $w \in W_i^\lambda$, then $w \models A$), i.e. for any freely determined $w_0 \models [s_i]\ A$. Further let us assume that for any freely determined i $w_0 \models [s_i]\ A$. Therefore, for any freely determined i $\forall w$ (if $w \in W_i^\lambda$, then $w \models A$), and since for any freely determined i $W_i^\lambda \neq \emptyset$, $\exists w\ (w \models A)$, and therefore also $\forall w\ (w \models A)$, and therefore finally $w_0 \models A$.

Let us further assume that $\exists w\ (w \models A)$ and $\exists w$ (not true that $w \models A$). Therefore there exists such w_1 and w_2 that $w_1 \models A$, and not true that $w_2 \models A$. Since for any freely determined i $\exists w\ (w \models [s_i]\ A)$ iff $\forall w\ (w \models [s_i]\ A)$, we have $w_1 \models A$ and $\sim w_1 \models [s_i]\ A$ or $\sim w_2 \models [s_i]\ A$ and $w_2 \models [s_i]\ A$.

In a freely determined situational model, the formula which is either a situational modal tautology or a situational modal countertautology, or finally a formula in the form of $[s_i]\ A$, is true in a certain possible world, if and only if it is true in all possible worlds. Furthermore, if both formulas A and B is possible in a certain world, then also the formulas in the following form $\sim A$, $A \wedge B$ and $A \rightarrow B$ are true in a certain possible world, if and only if they are true in all possible worlds. What is more, a freely determined formula A is true in all possible worlds, if and only if, for a freely determined i the following equivalent $[s_i]\ A \equiv\ A$ is a situational modal tautology.

We shall say that formula A is derivable from the set of formulas X, in symbols $X \vdash\ A$, if there exists such finite sub-set of set X $\{B_0, B_1, B_2, \ldots B_k\}$, that formula $(B_0 \wedge B_1 \wedge B_2 \wedge \ldots \wedge B_k) \rightarrow\ A$ is a proposition. We shall also say that the set of formulas X is inconsistent, if there is such formula A, that A and $\sim A$ are derivable from set X (or in other words: that formula $A \wedge \sim A$ is derivable from set X). We shall finally say that the set of formulas X is consistent, if it is not inconsistent.

The set of formulas X will be called maximally consistent, if X is consistent and if for any formula A, either A belongs to X or $\sim A$ belongs to X. According to Lindenbaum's lemma, each consistent set of formulas is a sub-set of some maximally consistent set of formulas.

If X is a maximally non-contradictory set of formulas, then for any freely determined formulas A and B,

$\sim A \in\ X$ iff it is not true that $A \in\ X$,

$A \wedge B \in\ X$ iff $A \in\ X$ and $B \in\ X$,

$A \rightarrow\ B \in\ X$ iff $A \in\ X$, then $B \in\ X$.

Conclusion 7. If the set of formulas X is consistent and formula $\sim A$ is not derivable from set X, then the set of formulas $X \cup \{A\}$ is consistent.

Let us assume that X is a consistent set of formulas. Moreover, $X \nvdash\ \sim A$. Let us assume indirectly that set $X \cup \{A\}$ is inconsistent. Thus, there exists such formula C that $X \vdash\ C \wedge \sim C$. Therefore, there exists such finite set $\{B_0, B_1, B_2, \ldots B_k\}$ that $\{B_0, B_1, B_2, \ldots B_k\} \subseteq\ X$ and formula $(B_0 \wedge B_1 \wedge B_2 \wedge \ldots \wedge B_k \wedge A) \rightarrow\ (C \wedge \sim C)$ is a proposition. Thus, formula $\sim (B_0 \wedge B_1 \wedge B_2 \wedge \ldots \wedge B_k \wedge A)$ and formula $(B_0 \wedge B_1 \wedge B_2 \wedge \ldots \wedge B_k \wedge A) \rightarrow\ \sim A$ are also propositions. Therefore, $X \vdash\ A$.

Conclusion 8. If the set of formulas X is consistent and formula A is derivable from set X, then set of formulas $X \cup \{A\}$ is consistent.

We shall say that formula A is a situational modal formula, when there exists such set X containing only formulas in the following form $[s_i]\ B$ or $\sim [s_i]\ B$ that A is derivable from X. We note that formula A is a situational modal formula, if and only if there exists such formula C in the form of $[s_i]\ B$ that implication C

$\rightarrow$ A is a proposition.

Let us assume that δ is a determined sequence of formulas in the following form: $[s_i]$ B. We will use A^{δ}_{n} to mark an n-th element in sequence δ. Let us define the following sequence of the sets of formulas

$$[X^{\delta}_{0}] = \mathbf{SLA}$$

$$[X^{\delta}_{n+1}] = \left| \begin{array}{l} X^{\delta}_{n} \cup \left\{A^{\delta}_{n}\right\},\ if\ \sim A^{\delta}_{n}\ is\ not\ derivable\ from\ X^{\delta}_{n}, \\ X^{\delta}_{n} \cup \left\{\sim A^{\delta}_{n}\right\},\ if\ \sim A^{\delta}_{n}\ is\ derivable\ from\ X^{\delta}_{n}. \end{array} \right.$$

Now let $X^{\delta} =_{DEF} \cup_n X^{\delta}_{n}$.

Let us note that:

(a) $\mathbf{SLA} \subseteq X^{\delta}$.

(b) For any n $X^{\delta}_{n} \subseteq X^{\delta}_{n+1}$.

(c) For any n set X^{δ}_{n} is consistent.

(d) X^{δ} is consistent.

(e) For any B either $[s_i]$ $B \in X^{\delta}$ or $\sim [s_i]$ $B \in X^{\delta}$.

(f) If A is a formula in the following form: $[s_i]$ B, but is neither a proposition nor a counterproposition, then for a certain δ_1 formula A, belongs to $X^{\delta 1}$ and for certain δ_2 formula $\sim A$ belongs to $X^{\delta 2}$.

Points (a) — (e) occur on the basis of the definition of the sequence of sets $\{X^{\delta}_{n}\}$, the definition of set X^{δ} and conclusions 7 and 8.

Let us assume that A is a formula in the form $[s_i]$ B, which is neither a proposition nor a counterproposition. Let us also assume that δ_1 is such a sequence of formulas in the following form $[s_i]$ B, that $A^{\delta 1}_{0} = A$. Obviously $A \in X^{\delta 1}_{1}$ and therefore $A \in X^{\delta 1}$. Further δ_2 shall be such a sequence of formulas in the form $[s_i]$ B, that $A^{\delta 2}_{0} = [s_j] \sim A$ and $A^{\delta 2}_{1} = A$. Since A is not a proposition, it cannot be derived from $X^{\delta 2}_{0}$. Yet, each equivalent in the form $A \equiv\ \sim [s_j] \sim A$ is a proposition and therefore also $\sim [s_j] \sim A$ is not derivable from $X^{\delta 2}_{0}$. Therefore $[s_j] \sim A \in X^{\delta 2}_{1}$. On the other hand each equivalent in the form of $[s_j] \sim A \equiv\ \sim A$ is a proposition and therefore $\sim A$ is derivable from $X^{\delta 2}_{1}$. So $\sim A \in X^{\delta 2}_{1}$ and therefore $\sim A \in X^{\delta 2}$.

For a determined sequence δ of formulas in the form of $[s_i]$ B, we shall now construct a situational model $\left\langle W^{\delta},\ \lambda^{\delta},\ V^{\delta}\right\rangle$.

W^{δ} shall be a set of all maximally non-contradictory over-sets of set X^{δ}. Symbols v^{δ} , w^{δ} ,... shall mean the elements of set W^{δ}. Certainly, $\exists w^{\delta}$ $([s_i] \in w^{\delta})$ if and only if $\forall w^{\delta}$ $([s_i] \in w^{\delta})$. $W^{\lambda\delta}_{i}$shall mean the i-th element of sequence λ^{δ}.

For any i let $W^{\lambda\delta}_{i} = \{v^{\delta} : \{B : \exists w^{\delta}\ ([s_i] \in B\ w^{\delta})\} \subseteq v^{\delta}\}$. We need to note that for any i set $\{B : \exists w^{\delta}\ ([s_i] \in B\ w^{\delta})\}$ is consistent. Let us assume indirectly that set $\{B : \exists w^{\delta}\ ([s_i] \in B\ w^{\delta})\}$ is inconsistent. Therefore there exists such finite sub-sets $\{B_0, B_1, B_2, ... B_k\}$ that formulas $B_0 \wedge B_1 \wedge B_2 \wedge ... \wedge B_k \rightarrow A$ and $B_0 \wedge B_1 \wedge B_2 \wedge ... \wedge B_k \rightarrow\ \sim A$ are propositions. Therefore formulas $[s_i]$ $B_0 \wedge$

$[s_i]\ B_1 \wedge ... \wedge [s_i]\ B_k \rightarrow\ [s_i]\ A$ and $[s_i]\ B_0 \wedge [s_i]\ B_1 \wedge ... \wedge [s_i]\ B_k \rightarrow\ [s_i] \sim A$ are also propositions and therefore they belong to every w^δ. Yet formulas $[s_i]\ B_0$, $[s_i]\ B_1$, $[s_i]\ B_2$,..., $[s_i]\ B_k$ also belong to every w^δ. Therefore formulas $[s_i]\ A$ and $[s_i] \sim A$, as well as $[s_i]\ A$ and $\sim [s_i]\ A$ also belong to every w^δ.

For every i set $W_i^{\lambda\delta}$ is therefore not empty.

Let us finally assume that for every i $V^\delta(P_i) = \{w^\delta : P_i \in\ w^\delta\}$.

Conclusion 9. For any freely determined formula A and any w^δ, $w^\delta \models A$ if and only if $A \in\ w^\delta$.

We will only demonstrate that for any freely determined formula A and any freely determined i, if for any w^δ $A \in\ w^\delta$ if and only if $w^\delta \models A$, then for any w^δ $[s_i]\ A \in\ w^\delta$ if and only if $w^\delta \models [s_i]\ A$.

Let us assume that A is such a formula that for any w^δ $A \in\ w^\delta$ if and only if $w^\delta \models A$.

Now, $[s_i]\ A \in\ w_0^\delta$.Therefore A $\in\ \{B : \exists w^\delta\ ([s_i] \in B\ w^\delta)\}$. Thus, if $v^\delta \in W_i^{\lambda\delta}$, i.e. $\{B : \exists w^\delta\ ([s_i] \in B\ w^\delta)\} \subseteq v^\delta$ then $A \in\ v^\delta$. Thus $\forall v^\delta\ (v^\delta \in W_i^{\lambda\delta} \rightarrow\ A \in\ v^\delta)$. Therefore $\forall v^\delta\ (v^\delta \in W_i^{\lambda\delta} \rightarrow\ v^\delta \models A)$. And so $w_0^\delta \models\ [s_i]\ A$.

Let us further assume that $w_0^\delta \models\ [s_i]\ A$. Therefore, $\forall v^\delta\ (v^\delta \in W_i^{\lambda\delta} \rightarrow\ v^\delta \models A)$, i.e. also $\forall v^\delta\ (v^\delta \in W_i^{\lambda\delta} \rightarrow\ A \in\ v^\delta)$. Therefore $\forall v^\delta\ (\{B : \exists w^\delta\ ([s_i] \in B\ w^\delta)\} \subseteq v^\delta) \rightarrow\ A \in\ v^\delta$. Set $X^\delta \cup \{B : \exists w^\delta\ ([s_i] \in B\ w^\delta)\} \cup \{\sim A\}$ is therefore contradictory. Therefore its finite subset $\{C_0,\ C_1,\ C_2,\ ...\ C_k, \sim A\}$ is also contradictory. Therefore formula $(C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k) \rightarrow\ A$ is a proposition. Therefore also formula $[s_i]\ C_0 \wedge [s_i]\ C_1 \wedge [s_i]\ C_2 \wedge ... \wedge [s_i]\ C_k \rightarrow\ [s_i]\ A$ is a proposition and belongs to w_0^δ. Yet, since $\{C_0,\ C_1,\ C_2,\ ...\ C_k\} \subseteq\ \{B : \exists w^\delta\ ([s_i] \in B\ w^\delta)\}$, each of the formulas $[s_i]\ C_0$, $[s_i]\ C_1$, $[s_i]\ C_2$, ... $[s_i]\ C_k$ belongs to w_0^δ. Therefore finally $[s_i]\ A \in w_0^\delta$.

Conclusion 10. If formula A is not a proposition, then there exists such situational model, in which formula A is not valid. Each situational modal tautology is therefore a proposition.

Let us assume that formula A is not a proposition, We shall demonstrate that for a certain sequence δ_n of formulas in the following form $[s_i]\ B$ formula A does not belong to a certain maximally consistent overset of set $X^{\delta n}$.

If A is a situational modal formula, then there exists such set X containing only formulas in the form of $[s_i]\ B$ or $\sim [s_i]\ B$, that formula A is derivable from X. Therefore, there exists such finite subsets of set X $\{C_0,\ C_1,\ C_2,\ ...\ C_k\}$, that formula $(C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k) \rightarrow\ A$ is a proposition. Obviously, conjunction $C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k$ is not a proposition. Since each of the formulas C_0, $C_1,\ C_2,\ ...\ C_k$ is in the form $[s_i]\ B$ or $\sim [s_i]\ B$, conjunction $C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k$ is equivalent to every formula in the form $[s_i]\ (C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k)$. Obviously, no such formula is a proposition. Therefore, each formula in the form $\sim [s_i]\ (C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k)$ belongs to a certain set in the form of

$X^{\delta n}$. Therefore formula $\sim (C_0 \wedge C_1 \wedge C_2 \wedge ... \wedge C_k)$ belongs to every maximally consistent overset of set $X^{\delta n}$. Set $X^{\delta n} \cup \{\sim A\}$ is therefore non-contradictory and therefore is a subset of a certain maximally consistent overset of set $X^{\delta n}$.

If formula A is not a situational modal formula, then it is not derivable from any set of formulas in the form of $[s_i]\ B$ or $\sim [s_i]\ B$. Therefore for any δ_n set $X^{\delta n} \cup \{\sim A\}$ is non-contradictory and is a subset of a certain maximally consistent overset of set $X^{\delta n}$.

Therefore, situational modal logic is set by the class of all situational models.

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