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LINGUISTICS, LOGIC AND THE LIAR PARADOX. COMMENTS ON THE ARTICLE BY A. GAWROŃSKI "THE 'LIAR SENTENCE' AS A RECURRING SENTENCE FUNCTION ('THE POLISH SOLUTION')"

Originally published as "Lingwistyka, logika i paradoks kłamcy. Uwagi o artykule A. Gawrońskiego <<Tzw. 'zdanie kłamcy' jako rekurencyjna funkcja zdaniowa ('the Polish solution')>>," Studia Semiotyczne 25 (2004), 67–75. Translated by Maja Wolsan.

There are many versions of the Liar Paradox (LP). J. Agassi names 13 of them (Agassi 1963: 237—238). But the most important one is related to Tarski's theorem that the truth predicate (P) is non-definable for systems that are sufficient for the formalisation of elementary arithmetic of natural numbers. Let S be such a system. We assume that S is consistent and that the syntax of S has been arithmetized as understood by Gödel. Let E be any sentence of S. E^* is the symbol of the Gödel number of E (these comments are a bit simplified as per: R. Smullyan 1992: 102—104). If formula $\mathrm{A}(\nu)$ belongs to the language of system S, then formula E is a constant point for formula $\mathrm{A}(\nu)$ if and only if $\mathrm{S} \vdash E \rightarrow \mathrm{A}(E^*)$. It can be proved that every formula $\mathrm{A}(\nu) \in \mathrm{S}$ has a constant point in S.

If P is a truth predicate (as defined by Tarski) for S, then for $\mathrm{S} \vdash \mathrm{P}(E) \rightarrow E$ for each sentence E [it is the so-called T-convention; in other words, formula E is a constant point for $\mathrm{P}(E)$]. However, predicate P (i.e. the set of all true sentences) is non-definable in S. Let us assume that it is. According to the constant point theorem, there exists formula E such that $\mathrm{S} \vdash E \rightarrow \neg\mathrm{P}(E^*)$. However, this leads to a contradiction, as we also have $\mathrm{S} \vdash E \rightarrow \mathrm{P}(E^*)$. What does formula $\neg\mathrm{P}(E^*)$ tell us? It tells us that a sentence with a Gödel number is not true, i.e. (remaining within the domain of classical logic) it is false. As the Gödel number of an expression can be considered its name, the above formula describes itself as false. Therefore, adding a truth-defining formula as an arithmetic axiom, I will get LP. Or by adding $\neg$P(E*) to axioms of S I will get a contradictory system. All this shows that the LP is not a toy or a curiosity, but a barrier to defining the arithmetic truth in arithmetic itself. Although the LP was formulated more than two thousand years before Tarski and Gödel, it has quite unexpectedly found an application in the deepest problems of elementary mathematics.

Alfred Gawroński would answer that what he is interested in is the LP in natural language, not in formal mathematical systems. Indeed, he discusses the status of the Liar Sentence (LS), i.e. a sentence claiming itself to be false, in everyday language. He claims that there is no LP, only an illusion of this antinomy, stemming from the wrong interpretation of the nature of the LS. I will discuss the arguments supporting this thesis later in this article. For now, we need to identify the object of this dispute. Although the natural language cannot be subjected to arithmetization, it can be ordered a little, in particular to eliminate the obvious incidentality of the LP from the original sentence:

(1) This sentence is false.

In particular, we can number natural language sentences and put them in countable strings like $C = Z_1, Z_2, Z_3 \dots$. This way k, being a number in subscript Z_k, clearly defines the place of $Z_k$in string C. Let us now introduce the following convention: (k) means sentence in the k-th position in C. Instead of speaking about sentences, we can therefore speak about their numbers such as (k). They are the imitations of numerals in the language of arithmetic. There are as many strings C as the potential number of orderings of the sets of sentences, that is always $2^n$for a set composed of n sentences. Analysing the possible strings, we will finally find a string such that:

(2) (k) $= Z_k$is false.

Let us now discuss the sentence '$Z_k$is false'. If it is true, then (k) is also true. But (k) is sentence Z_k. It follows that:

(3) $Z_k \rightarrow Z_k$is false,

and further through the T-convention (Z is true if and only if Z)

(4) Z_k is true $\rightarrow$ $Z_k$is false.

If the sentence '$Z_k$is false' is false, then (k) is also false and Z_k is true, which again leads to (4).

In fact, the reasoning for natural language repeats the basic elements of mathematical argumentation. Both also show the key importance of T-convention and T-equivalence in the derivation of the LP, and consequently show that the proper definition of truth as understood by Tarski is impossible for natural language as a whole. Let me add some additional comments. For each (k) we can build an LP according to the following pattern:

(k) The sentence written in line (k) is false.

At first, it seems that it is the same nonsense as in (1), implying the need to adopt nonsensical equivalence:

(5) This sentence is false if and only if this sentence,

(6) (6) is false if and only if (6).

This seeming nonsense disappears when we realise that these are sentences which, under certain conventions, are introduced by the expressions 'this sentence' and '(6)'. Furthermore, the derivation of contradiction does not imply marking the sentences expressing the LP as either true or false. The contradiction arises in both cases. We do not, therefore, need to wonder what the LS actually states, we only need to examine what it expresses.

I cannot analyse all of Gawroński's theses here, but I will try to comment on the most important ones and to prove his thesis that the LP is illusory to be an illusion itself. I will discuss the following issues: (a) the concept of meta-sentence and theme-rheme structure of sentences; (b) the syntactic ambiguity of expressions such as 'this' and '(k)' in sentence (1) and convention (k); (c) the problems of self-reference of sentences.

Re. (a), without going into general definitions, I will just focus on an example of a meta-sentence, namely sentence (6). It consists of a propositional predicate 'is false' and its argumentation, i.e. '(6)'. However, this sentence may be called pathological, as the argument in a normal meta-sentence would have an object argument, e.g.:

(7) Sentence *Z* is false.

Generally, each normal meta-sentence has its rheme, i.e. the sentence in which we speak, and theme, i.e. the sentence of which we speak. In this particular case, (7) is a rheme as a whole and sentence *Z* is the corresponding theme. Rheme is grammatically more important than theme, and Gawroński states as much in his text. On the other hand, he says that sentences have a theme-rheme structure. But if the rheme is a sentence in which we speak and theme is the sentence of which we speak, then instead of a single sentence we have an ordered pair <rheme, theme> (this direction seems right due to the said order of importance) composed of two sentences. Another way to understand rheme and theme, more consistent with the need to analyse the structure of sentences in these categories, is to treat 'is false' in (7) as a rheme and sentence *Z* as a theme. The rheme would thus be a sentential connective of a sentential argument, superordinate to it. This analysis is more suitable for forms such as:

(8) It is false that *Z*.

then for sentences like (7), the rheme of which is the relevant predicate (e.g. 'is true' or other), and the argument being not a sentence but rather its name. The difference between (7) and (8) is not very important for further discussion, thus I am going to use sentences of type (7). However, I do not know what to do with theme and rheme. I will proceed just as if both methods led to the same consequences.

I am going back to Gawroński's article, although I will not always use his own terminology. A normal, non-pathological meta-sentence requires closure/complementation by an object-sentence, e.g. 'Snow is black.' But, in fact, all known versions of the LP operate in meta-sentences that do not end with object-sentences. Thus, sentences created according to convention (k) should not be considered correct, as they violate the basic syntactic rule for meta-sentences, i.e. that a correct meta-sentence ends with an object-sentence. How I understand it is that this superordination of rheme over theme consists in the meta-sentence having an object argument.

The key problem is to find an answer to the question whether predicate arguments (from now on I will omit logical operators) are to be limited. It is where the real dispute begins. Gawroński claims that even everyday language forces some restrictions, like the need to complement a meta-sentence with an object-sentence. I do not believe it is so. Let us consider

(9) (k) and (k) are equivalent.

This is a typical meta-sentence, which is not controversial from the perspective of an everyday language. However, it does not have an object complement — nor does it need to. I also do not see the reason to claim, like Gawroński does, that self-complements, i.e. situations when a meta-sentence complements itself, must be excluded *a priori* as absurd. This decision is completely arbitrary. After all, Epimenides, the stoics or Savonarola were competent users of their own mother tongues and invented relevant LSs as absolutely acceptable — though perhaps a bit odd — examples of sentences in the grammatical sense.

Gawroński clearly confuses the syntactic and semantic orders. For a logician, the fact that theme is subordinate to rheme is a banality and means simply that a logical connective is defined by what it creates and of what it creates. In this sense, the argument of the rheme related to (7) is the name of a sentence (or a sentence itself, if we are considering other structures or pairs such as <rheme, theme>). Gawroński adds a new requirement, namely that it must be an object-sentence. This is a semantic argument, as object-sentences are defined in semantics, not in syntax. From the point of view of syntax, this condition is arbitrary. Gawroński continues by saying that there exists no LP, that the structure of meta-sentences of the type derived from convention (k) were just wrongly recognised. As I have shown, however, it is not about structure, but about semantics. A logician would therefore claim that the paradox indeed exists, without assuming any syntactic constraints, and then would conduct a relevant reasoning (which, interestingly, is of no interest to Gawroński) and propose certain restrictions. All in all, these restrictions are not very far from what Gawroński proposes. Tarski's solution consisted in assuming that if a meta-sentence predicate belongs to k-order language, then it concerns sentences of k-1 order, although the whole meta-sentence must be formulated in the former language. Both these positions can finally be reconciled by assuming that a non-pathological meta-sentence must be such that its rheme is one step higher than the theme. This way syntax is reconciled with semantics.

In addition, I should mention that the sentence:

(10) Sentence (10) is true

does not lead to any problems (at least in as much as we operate the standard concept of logical consequence (see Woleński 1993: 89—102, for the Truth-Teller Paradox), although it is also wrong, just as the sentences based on convention (k). This fact is an additional argument supporting the view that linguistic criteria of accepting sentences as grammatically correct, in particular those based on thematic-rhematic analysis, are insufficient for logic.

Re. (b), according to Gawroński, one of the sins of logical analysis of the LP is related to the following construction:

(11) Expression '(k)' means 'sentence (k) is false'.

Gawroński says that this way expression '(k)' functions syntactically in two meanings. Although both instances of '(k)' refer to the same, i.e. to the sentence marked as '(k)', they function in different syntactic forms, as the second instance means an example of sentence (k) which is subordinate to the one marked by the first instance. This is what, according to Gawroński, is ignored by logicians.

First, we should observe that (11) expresses only that there exists such a numeration of sentences that sentence number (k) is 'Sentence (k) is false.' Even if from a linguistic point of view it is indeed as Gawroński says, i.e. that symbol (k) stands for a specimen of a sentence, subordinate to another specimen, this fact is essentially of no importance to the subject issue. Let us notice, by the way, that a new understanding of subordination (and its opposite, superordination) has appeared, that is the relation between the specimens of sentences instead of their rhemes and themes. Gawroński does not stop at (11), he also discusses the LS from the same point of view. He thus writes (Gawroński 2004: 49):

> Sentence expressions such as 'This sentence is false' or '*A* is false' (as the result of the assumption that '*A* means "*A* is false"') already in the assumption contain *A* in two different syntactic positions which are NON-REDUCIBLE to each other. [...] For if they were, we would have one and the same specimen of the expression, superordinate to itself, which is a syntactic absurd.

Nevertheless, the names 'this sentence' and '*A*' appear in the quoted sentences only once and it is not very clear that they have different syntactic positions. Maybe what Gawroński means is rather that in LP derivations expression '(k)' acts sometimes as a name, and sometimes as a sentence. However, as obvious as it is, it does not imply that we are dealing with one and the same specimen of a given expression, e.g. 'This sentence is false,' nor does it imply that we are trying to reduce one to the other. It is, in fact, quite the opposite [cf. the comments to (5) and (6)), as the logical analysis of the LP clearly recommends a careful distinction between '(k)' as a name and as a sentence.

Second, logicians do not ignore anything in this respect. Since the time of Leśniewski and Tarski, they have been pointing out that the lack of distinction between expressions and their names entails serious semantic problems. Consequently, a symbol introduced to mark a sentence may be interpreted both as its name and as the sentence itself. This leads to complications, as self-names appear in the context of semantic terms.

Re. (c), Gawroński believes that in some cases the self-reference of sentences is not a big problem, just as in the following case:

(13) This sentence (i.e. sentence (13)) is composed of seven words.

It is true, which can be easily verified by counting the elements. Gawroński claims that it is correct from the perspective of the theme-rheme structure — as opposed to the LS. In fact, (13) is an elliptical abridged version of the following sentence:

(14) 'This sentence is composed of eight words' contains seven words,

where 'this' refers to (13). This, however, leads to a disastrous consequence. If (13) is an abridged version of (14), we have

(15) (13) $\rightarrow$ (14).

On the other hand, the expression 'this sentence' in (14) means the same as number (13). From this, it follows that:

(16) Sentence number (13) is composed of seven words if and only if 'sentence number (13)' is composed of seven words.

The equivalence in (16) is false, as its right side is true, whereas its left side is false; the expression 'sentence number (13)' contains three words. More technically speaking, Gawroński made a groundless assumption that (13) is a constant point for (14).

In fact, Gawroński uses the method of analysis of (13) and (14) only as an introduction to his critical remarks on self-reference of LS-type metasentences. One of the arguments against the self-reference of such sentences is the syntactical ambiguity of the methods of identification such as 'sentence number (k)'. We have already covered this. Gawroński's disquisition on this does not seem conclusive, therefore, I will proceed to the next argument, which is that the LS has no self-reference but instead has recurrence, i.e. generating a string of utterances repeating the first step. We start with (k), then we add '(k) is false', then "'(k) is false" is false', etc. As a result, we get a string

(⊢) <(k), (k) is false, '(k) is false' is false, "'(k) is false" is false' is false, . . . > ,

in which each subsequent specimen of '(k) is false' is subordinate to the previous one, and the previous one is a rhematic negation of the next one. The even-numbered formulae have the same logical values, and the odd-numbered formulae have opposite values. Thus, there is no self-reference in LSs, there is only recurrence. It is not strange that the logical values of various specimen of the LS in string (⊢) cyclically change from even to odd, and it is not a paradox either. This, according to Gawroński, explains the illusion of the LP.

But this all is an illusion itself that the LP has thus been annihilated. Gawroński thinks that string (⊢) is the same as the string:

(⊢⊢) <sentence (k) is false, sentence (k) is false, (k) is false. . . >.

In a sense, it is indeed so. We replace (k), i.e. the first element in string (⊢), and this way we get the first element of (⊢⊢). Then we repeat this operation in the other direction, thus getting the second element of (⊢⊢), etc. Two strings are identical if and only if their subsequent elements are identical. In this case, as we are dealing with sentences, we say that two strings of sentences are equivalent if and only if their subsequent elements are logically equivalent. Let us look, then, at the third element of string (⊢) and the third element of (⊢⊢), i.e. "'(k) is false" is false' and '(k) is false'. The first sentence is equivalent to '(k) is true', which gives us another instance of the LP because (4). The paradox can be formulated for any corresponding elements of the two strings. If we only look at (⊢⊢), the LP occurs for each of its elements. Consequently, recurrence does not eliminate self-reference nor the LP.

Gawroński might comment that the above analysis fails to take into account subordination, superordination, rheme and theme. I will now prove that taking these concepts into account does not lead to the conclusions drawn by Gawroński. Let us assume that string (⊢⊢) is generated in the following way: We start with '(k) is false'. Regardless of the fact whether the rheme is the whole sentence or the predicate 'is false', there is a need to add a relevant argument, which again is '(k)' (or a sentence marked by the symbol (k)). Again, we insert '(k) is false'. This procedure can be repeated any number of times and thus (⊢⊢) is created. Let us now assume that each previous element of the string is a negation of the next one and that the values of the elements change in the following way — even numbers are false, odd numbers are true. We translate (⊢⊢) into

(⊢⊢⊢) $<e_1,\ e_2,\ e_3,\ \ldots>$,

assuming that $e_i \rightarrow\ e_j$, where i, j are pairs of odd or even elements, while $e_i \rightarrow\ \neg e_j$where one of the indicators is even and the other is odd. However, string (⊢⊢) is diametrically different from (⊢⊢⊢), as in the first one all elements are equivalent as identical (as Gawroński defines them himself). Therefore, we cannot say that the recurrence of the LS generates a string of identical specimen of the sentence '(k) is false', if at the same time we assume that previous elements are negations of the next ones. This assumption generates a string (it still applies that even elements are true and odd are false; we also assume that '(k) is false' is true if and only if (k) is false):

(⊢⊢⊢⊢) <(k) is false, (k) is true, (k) is false . . . >.

The string concerns one specific sentence, marked as (k). It was generated in accordance with the sentence's internal structure, and not by automatically alternating any sentence and its negation, thus it reflects the antinominal nature of sentence (k). It is unquestionable that self-reference plays a key role here. I would like to underline that Gawroński ignores the delicate problem of negation of the LS. The negation of (k) must be a sentence numbered at least (k+1), and therefore cannot produce a specimen of a sentence identical with sentence (k). Simply speaking: the negation of 'This sentence is false' is not the said sentence.

Gawroński completely failed to take into account the fundamental difference between the self-reference of type (13) and the one related to semantic concepts. Each version of the LP uses, indirectly (as in the Circle of

Liars/Vicious Circle Principle) or directly (as in the one-sentence version), the fact that T-equivalences determine constant points for sentences such as 'Z is true' and thus ensure compression of such formulae to their arguments or expansion of the arguments to expressions with a truth predicate. This two-way operation shows that any analysis in terms of subordination—supraordination or rheme—theme is secondary in this case to intentional contexts, e.g. 'X believes that Z.' Consequently, self-reference of sentences with semantic predicates and T-equivalences for such sentences are a source of LP, both in formal languages and in natural language. If we remain in the domain of classical logic, we can either prohibit formulating T-equivalence for sentences expressing the LP or eliminate self-reference.

There was a time when logicians thought themselves the only people competent to talk about any language, including natural language. They claimed, for example, that the logical grammar of natural language is the same as of a formal system. It is, fortunately, all in the past. Gawroński, on the other hand, presents the opposite extreme, or at least something close to it. He wants, namely, logical analyses to meet linguistic requirements. But a logician cannot be constrained by the view that sentences have a theme-rheme structure, even if this view is currently commonly accepted. One hundred years ago it was not, and in the next one hundred years yet a different theory might prevail. I do not think that the structures accepted in propositional calculus or predicate calculus depend on what linguists think about the nature of sentences. It may be that for a linguist theme is always subordinate to rheme, but for a logician it is not the case in extensional contexts, or at least it does not always have to be so. A logician would say that the relation of syntactic equivalence is a particular type of subordination, just as being the same set is a particular type of a subset. It is true that the distinction between expression-type and expression-specimen is important, but it cannot determine the correctness of a reasoning based on simple logical rules, just as the nature of representation or of the carrier of truth.

The latter remarks do not suggest that logic and linguistics should be separated. Indeed, the disqualification of the LS as incorrectly constructed because of the mixing of levels of language corresponds to the admission that the theme-rheme structure of such a sentence is pathological. The indication that the LS produces a string without a terminal element is a very interesting symptom of the defectiveness of this sentence. But this fact does not prove the non-existence of the LP, rather it proves the paradox real in situations when the rules of language levels or theme-rheme structures are violated.

Gawroński, however, questions this common point. In this article, I have tried to show that he is not successful in this. Let ’the Polish solution’ continue to be associated with Tarski. No persuasive comments on logicians — that they do not understand this or that as regards natural language, complicate the LP, propose ever new *ad hoc* solutions, or treat important linguistic questions such as the subordination relationship as absurd because they are not familiar with the structuralism culture of contemporary linguistics — can change it. They are examples of wishful thinking, not arguments.

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