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THE ISSUE OF LINGUISTIC AMBIGUITIES IN WITTGENSTEIN'S SECOND PHILOSOPHY AND IN SCHÄCHTER'S CRITICAL GRAMMAR

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1. The formalist turn in Wittgenstein's philosophy and its consequences

1.1. The formalist turn in Wittgenstein's philosophy

By a seemingly strange twist of fate, the formalist approach to language became popular after Hilbert's project of providing a formalist foundation for mathematics foundered on Gödel's famous incompleteness theorems. Even in September 1930, during the congress in Königsberg, where Gödel presented his results for the first time, Carnap still defended logicism against the intuitionist and formalist views on the foundations of mathematics. Furthermore in linguistics, the formalist approach to syntactic issues had only become disseminated in the 1930s, due to the distributionists of Bloomfield's school, and found its paramount expression in Hjelmslev's *Prolegomena to a Theory of Language* (1953): according to Hjelmslev, both the linguistic theory and the grammar of a given language are nothing more than a calculus. Among the philosophers and logicians of the Warsaw school, under the influence of Hilbert's works, the impact of formalism had already grown stronger in the late 1920s and resulted in the invention of metalogic. At around the same time, after a 'linguistic turn' — and, in a sense, within its limits — Wittgenstein's philosophical reflection on language and the foundations of mathematics also gravitated towards formalism and its methods. In the years 1928—1930, in an attempt to clarify fundamental ideas of the *Tractatus* in a conversation with Schlick and Waismann, Wittgenstein turned to the formalist account of language. His reception of formalism and its role in forging his 'second philosophy' is clearly illustrated by a remark made on 19 June 1939 in the presence of Schlick and Waismann:

Part of formalism is right and part is wrong. The truth in formalism is that *every syntax can be conceived of as a system of rules of a game.* I have been thinking about what Weyl may mean when he says that a formalist conceives of the *axioms of mathematics as like chess-rules.* [cf. Weyl 1927a: 25] *I want to say that not only the axioms of mathematics but all syntax is arbitrary.* In Cambridge I have been asked whether I believe that mathematics is about strokes of ink on paper. To this I reply that it is so in just the sense in which chess is about wooden figures. For chess does not consist in pushing wooden figures on wood. [. . .] It does not matter what a pawn looks like. It is rather the totality of rules of a game that yields the logical position of a pawn. A pawn is a variable, just like 'x' in logic. (Waismann 1979: 103—104; my emphasis)

The passage, although illuminating, requires some explaining — not unlike many analogous statements made by Wittgenstein in the 1930s. Note, first of all, that Wittgenstein refers not so much to the state-of-the-art formalism of Hilbert's school as to the so-called older formalism of Heine, Thomae, and — strange though it sounds — Frege, who developed the formalist account of the foundations of mathematics outlined by Heine and Thomae into a robust alternative to logicism (albeit for purely critical reasons). It is from the second volume of Frege's *Grundgesetze der Arithmetik* (1903: §§ 87—130) that Wittgenstein took the central idea of the older formalism — the notion of 'sign-game' (*Zeichenspiel*), as well as 'calculus-game' (*Rechenspiel*),[1] which he then transformed into the cornerstone of his second philosophy — the concept of language-game. The decisive step on the way from the account of arithmetic as a sign-game to the idea of language-games consisted in expanding the concept of sign-game into 'sign-games with elements of reality' or, to be more precise, games in which some elements of reality such as the metre standard (yardstick), colour samples, and the like, are used as means of representation.[2]

[1]Cf. esp. Thomae 1898: 1—10 and Thomae 1906 — the latter article was a response to Frege's critique of the position presented in the former work; it begins with the following remark, which distinctly reveals the affinity of Wittgenstein's idea of language games with the older formalism: "a person who wishes to ground arithmetic in a formal theory of numbers, that is, in a theory which does not ask what numbers are and what they mean but merely what we need of them, will gladly consider what I believe is another example of a purely formal creation of the human intellect, namely — chess. Chess pieces are signs that have no other content inside the game but the one imposed on them by the game rules."

[2]This expansion, as suggested by Waismann's notes and numerous passages from *Philosophical Remarks*, was probably carried out around 1930, see esp. Wittgenstein

1.2. Some consequences of the formalist turn

1.2.1. The arbitrary nature of the linguistic sign

From a formalist point of view, each grammar consists of rules for the use of signs — rules that might as well be entirely different. In this sense every grammar is arbitrary or conventional and — as such — cannot be justified. The same is true of meaning, defined in the most general terms by Wittgenstein, followed by Schächter, as the totality of syntactic rules determining the 'use' of signs. On 19 June 1930, Wittgenstein explained this 'formalist' platitude — criticized and rejected by Frege — to Waismann in the following way:

We can lay down the syntax of a language without knowing if this syntax can ever be applied. (Hypercomplex numbers.) All you can say is that syntax can be applied only to what it can be applied to. [. . .] The essential thing is *that syntax cannot be justified by means of language*. When I am painting a portrait of you [Waismann] and I paint a black moustache, then I can answer to your question as to why I am doing it: Have a look! There you see a black moustache. But if you ask me why I use a syntax, I cannot point at anything as a justification. *You cannot give reasons for syntax. Hence it is arbitrary.* Detached from its application and considered by itself it is a game, just like chess. This is where formalism is right. (Waismann 1979: 104—105, my emphasis)

It immediately springs to mind, of course, that there are numerous analogies with the well-known linguistic principle of the arbitrary nature of the linguistic sign. For this reason, it must be stressed that the above principle of the arbitrariness of syntax, on Wittgenstein's and Schächter's construal, has much more serious consequences for semantics than the principle of the arbitrary nature of the sign — long-known in comparative linguistics and made the basis of synchronic linguistics by de Saussure. De Saussure's principle of the arbitrary nature of the sign concerns, above all, the choice of phonic and graphic material used to designate given concepts and —indirectly — things, their properties, and states of affairs (de Saussure 1959: 67—70). Even if we accept de Saussure's thesis about the synchrony of the division (articulation) of the spoken chain and the chain of concepts (de Saussure 1959: 111—113), it leaves open the question whether designata of linguistic signs also fall under the principle of arbitrariness. Slotty, who was the first member of the Prague Linguistic Circle to supplement synchronic linguistics with semantics, was inclined to believe that designata do not depend on language and its arbitrary, synchronic delimitations of units within the spoken and mental chains. In fact, by invoking hypotheses regarding the origin of language,

1981, 1964: §§ 38—48.

Slotty denied that meaning (*Meinung*) is dependent on the grammatical and lexical structure of language; rather, it is entirely autonomous — 'logical' (Slotty 1929: 99).[3]

Such a distinction is entirely alien to Wittgenstein's second philosophy and to Schächter critical grammar, and so the 'principle of conventionality' is much more unequivocal in them. *All* rules determining usage, and thereby meaning, are arbitrary or conventional — both the rules for the use of signs within the 'sign-game' of a given language and the rules for interpreting the game or, more generally, for its application to describing reality, giving orders, carrying them out, etc.

1.2.2. From the 'ordinary' grammar to the philosophical grammar

In the first chapter of *Prolegomena to a Critical Grammar*, Schächter shows that from a semantic viewpoint there is no relevant difference between 'social use', or rather 'tacit convention' (*stillschweigende Festsetzung*), and explicit convention, *willkürliche Festsetzung* (1973: 8—9 [part 1, chap. 1, §5]). More precisely, the way of laying down or establishing rules of a language is not relevant to the meaning itself. This leads to a significant expansion of the scope of the critical grammar — which focuses on 'essential rules', i.e. rules that differentiate meanings of signs — in comparison with traditional linguistics. In the light of the critical grammar, there is no significant difference between the so-called natural languages and the so-called artificial languages (or simply calculi). Likewise, there is no difference between them and languages that could hardly be classified either as 'natural' or 'artificial', such as the language of chemistry, physics, sociology, and other branches of science.

From the viewpoint of the older formalism assumed by Wittgenstein, followed by Schächter, the principle of arbitrariness of linguistic rules applies even to the so-called 'logical rules of language' and especially to the rules of the classical propositional calculus and the classical predicate calculus.[4] This means that both the 'logic of the content' (Wittgenstein 1974: 217) and the 'philosophical

[3]Slotty's answer to the question about the synchrony of the articulation of speech and thought and the resulting dependence of thought on language is as follows: "This question — already on account of pre-linguistic assumptions — must be answered in the negative; for thought and speech are not completely correspondent in the sense that each thought-category should correspond to a unique formally defined category of words." Thus not every kind of words and affixes entails a difference in denotation (*Meinung*). It is so because thought-categories, the subject matter of semasiology, are understood here as semantic categories in the logical sense (*Meinung*), i.e. they apply to denotation instead of connotation.

[4]In fact, it was not Hilbert, Wittgenstein, or Carnap in the *Logical Syntax of Language* (1937), but already Frege in the second volume of *Grundgesetze* (§ 90), who declared that: "it is quite true that we could have introduced our rules of inference

grammar'[5] are devoid of any 'logical space' of meanings. This in turn makes for the radical separation, typical of the state-of-the-art formalism, of the calculus or — as Wittgenstein would put it — the pure 'sign-game' of language, from its possible 'external' applications and above all from its semantic interpretation. Even so, it must not be overlooked that for Wittgenstein and Schächter, in contrast to Hilbert and his school, these rules of the 'pure sign-game' (*bloßes Zeichenspiel*) determine the meanings of signs, and so they are semantic in character.

Schächter, in his critical grammar, which was supposed to be a generalized 'grammar of meaning' — encompassing non-natural languages as well — clearly distinguishes, like Wittgenstein, inessential rules concerning the choice of the 'material of the sign' from rules defining the use of signs — signs, which can be rendered or expressed by means of any material: wooden figures, written marks, sounds, or mental and physiological processes. In the formalist framework of Wittgenstein's 'philosophical grammar' and Schächter's 'critical grammar', we must disregard rules for the use of signs which fail to 'affect the gameplay' of sign-games, including sign-games with elements of reality, or rules that — to use Wittgenstein's terminology from the period of *Philosophical Remarks* and *Philosophical Grammar* — fail to bear on 'mathematical diversity of language'.[6] The distinction between two kinds of linguistic rules became the basis for the distinction between the 'grammar of material', also called the 'ordinary grammar' by Wittgenstein, and the critical grammar or the grammar of meaning, called

and the other laws of *Begriffsschrift* as *arbitrary stipulations*, without speaking of the reference and the sense of signs. We would have then been treating the signs as *figures*. What we took to be the external representation of an inference would then be comparable to *a move in chess*, merely *the transition from one configuration to another*" (Geach and Black 1960: 185—186, my emphasis).

[5]This is how Wittgenstein described the philosophical views set out in the *Big Typescript*, whose part has been edited by Rush Rees under the title *Philosophische Grammatik* (intended by Wittgenstein) (Wittgenstein 1984, 1974). Schächter, on the other hand, often calls his critical grammar, concerned with the rules of language which affect meaning, the grammar of meaning.

[6]The distinction between essential and unessential rules of language, central to the critical grammar, was explained by Schächter in the following way: "(I) Suppose the rules of chess included the specification that the fields must be squares and either black or white. If now we play a game and notice that these rules do not affect the game so that any position on such a board may be translated to one with rectangular fields that are red or yellow, we say that these rules are inessential. (II). There are rules that state that a pawn reaching the eighth rank may be exchanged for any piece of the same colour except the king, or that castling is subject to precise and defined conditions: these rules we denote as essential, for without them different positions from the usual ones could occur on the board. These rules belong to the ŕmeaningő of the pieces, just as the rule that pawns move straight and take diagonally" (Schächter 1973: 21 [part 1, chap. 3, § 2]).

critical by Schächter precisely in order to differentiate it from linguistics, which focused on rules regarding the material of signs and lacked a clear-cut distinction between these two kinds of rules. The account of language as a sign-game established by its 'essential rules' is closely connected with the 'formalist' idea — typical of Wittgenstein's second philosophy and Schächter's critical grammar — of the sign as a type of figure, whose meaning amounts to its 'use in the game' and so is equally arbitrary as the rules that define it.[7]

For present purposes — in the context of our discussion of linguistic ambiguities— the crucial aspect of Wittgenstein's reception of formalism is its relation to the purely descriptive project of semantic reconstruction, as opposed to the project of providing a foundation for mathematical theories (as is the case with Hilbert's formalism). It found its paramount expression in the *Blue Book*, where Wittgenstein put the fundamental methodological principle of his new philosophy in the following way:

I want to say here that it can never be our job to reduce anything to anything, or to explain anything. Philosophy really *is* 'purely descriptive'. (Wittgenstein 1958: 18)

The purpose of reconstructing the 'sign-game', or simply the calculus, of a language, was not to legitimize some theory or even to test the consistency and decidability of a deductive system; the aim was to reconstruct the grammar of meaning of a given language. Thus, for Wittgenstein and Schächter, the formalist methods of scrutinizing language — in the spirit of the older formalism — did not serve as a tool for a logical critique of certain theories but rather were part of the descriptive semantics, whose main task is to offer an accurate account of semantic and grammatical characteristics of a language under investigation and to explain its distinctive nature. For this reason, Wittgenstein and Schächter ruled out, on principle, such operations— crucial for the formalism of Hilbert's school — as the translation of a given theory or its language into the language of formal logic or — after 1930 — arithmetization of syntax.

Wittgenstein's descriptivist approach is closely associated with his critique of the *Tractatus* and with the underlying acknowledgement of the limitations of the symbolism of *Principia Mathematica*, which Wittgenstein had *a priori* regarded as universally applicable and sufficient for reconstructing and expressing all possible contents (Fregean senses). From the perspective of general descriptive

[7]In the above-mentioned conversation with Waismann and Schlick, on 19 June 1930, Wittgenstein invoked the example of chess to explain the notion of meaning that springs from treating language as a sign-game: "*the signs can be used the way they are in the game*. If here (in chess) you wanted to talk of őmeaningŕ, the most natural thing to say would be that *the meaning of chess is what all games of chess have in common*" (Waismann 1979: 105, my emphasis).

semantics, the logic of the 'subject-predicate form' (Waismann 1979: 46—47) and the grammar of logical connectives (Wittgenstein 1964: 109—110 [§ 82]), turned out to be just a part of the grammar that determines the meanings of 'our language'. In order to distinguish the latter from the traditional grammar, Wittgenstein dubbed it 'the logic of the content' and contrasted with 'the logic of the form':

Discuss: The distinction between the logic of the content and the logic of the propositional form in general [*Logik der Satzform überhaupt*]. The former seems, so to speak, brightly coloured, and the latter plain; the former seems to be concerned with what the picture represents, the latter to be a characteristic of the pictorial form like a frame. (Wittgenstein 1974: 217)

For Schächter, it is already clear that questions about the logical properties of language and its signs can only be answered within the framework of the critical grammar. After all, their essential logical properties are nothing but semantically essential language rules, which can, and must, be established, or rather 'read off', on the basis of the actual use of signs. Naturally, as such, they are limited to a given language, and the issue whether there are logical rules that are common to all languages, or even necessary, boils down to the question about the so-called grammatical universals — understood, of course, in terms of the critical grammar (see Schächter 1973: 60—64 [part 2, ch. 1, §§ 5—7]).

2. Limitations of the formalist account

Despite numerous fruitful applications and the undeniable progress, the formalist point of view and the formalist methods associated with it imposed certain constraints on logical, philosophical, and — at least since 1959 — linguistic studies of language. The limitations of the modern-day Hilbertian formalism, concerning the interpretation of calculi and their application in the proofs of consistency, decidability, completeness, and soundness of the reconstructed theories, are well-known and acknowledged. Yet equally important for reconstructing semantics are some limitations of the formalist methods hidden in the interface between three parts or stages of linguistic research distinguished by Carnap (1939: 3—29): pragmatics of the language, its 'pure' semantics, and the 'calculus' built on it. In accordance with the method of reconstructing the calculus of a language, only semantic relations that fall under exact, unambiguous rules can be transferred to the semantics and the calculus of the language and reflected in them. This imposes fairly restrictive limits on a formalistically understood reconstruction of language, which persuaded Tarski to confine the applicability of his concept of truth to formalized languages of deductive sciences.

Even though habits and conventions constituting the semantics of a language, need not be consistent in order to be expressible in the calculus, they must not be polysemous, vague, or simply unclear. The corresponding linguistic ambiguities cannot be smuggled into the calculus (into the 'pure' grammar — encompassing both the 'surface' and the 'deep' structure) of the reconstructed language and adequately represented. This kind of ambiguity, common not only in natural languages but also in the languages of particular branches of science, are *a limine*, methodologically, so to speak, doomed to remain at the threshold of pragmatics (Carnap 1939: 11—12). For this reason, Eleonor Rosch (1978) and other mentalists argue that logic, or even 'the tradition of Western reason' (Rosch 1978: 35), is fundamentally incapable of adequately reflecting the systematic vagueness and ambiguity of concepts and sentences of natural-language sentences. We frequently hear that language is no formal system, no calculus, and cannot be described in logical terms; and in justifying such assertions, mentalists often appeal to Wittgenstein and his 'second philosophy' (e.g. Rosch 1978: 36, Lakoff and Johnson 1980: 71—76, 122—125, 162—182, Taylor 1989: 38—40). Are they right, however, in regarding Wittgenstein as the precursor of such views?

3. (G) Grammatical ambiguities

Wittgenstein's conception of the sign as the totality of rules for its use together with the notion of sign-games and language-games lets us analyse possible kinds of vagueness and ambiguity on several planes. First and foremost, we should ask whether a simple sign of a language can be ambiguous or vague at all. If we set aside its material side and focus, like Wittgenstein and Schächter, on the rules of use, it seems that semantic ambiguity can only consist in the rules of use being (1) ambiguous, i.e. not uniquely specified or fuzzy, or (2) inconsistent. Yet, in the first case, can we speak of grammar, and especially of grammatically determined ambiguity?

There are, of course — as Wittgenstein explicitly acknowledges — games and languages "without their rules being codified" (Wittgenstein 1974: 63 [§ 26]). In assuming this, however, Wittgenstein declares that "we look at games and language under the guise of a game played according to [unambiguous] rules. That is, we are always comparing language with a procedure of that kind" (Wittgenstein 1974: 63 [§ 26], my addition). Methodological reasons justifying such an approach to the issue of ambiguity seem obvious. Most importantly, only a comparison with a calculus *sensu stricto* allows us to recognize and specify the lack of precision of relevant rules for the use of signs in a given game or a given language. Secondly, only a comparison with a calculus lets us formulate, in an exact way, the question about the specific nature of ambiguity in a given case and about its scope in a given game or language. Thus Wittgenstein's approach to the problem of ambiguity of concepts and propositions, as well as Schächter's parallel position, are at odds with the account offered by the advocates of the theory of prototypes.

3.1. (I) Inconsistency in the rules of use

The most obvious kind of grammatically determined ambiguity is connected with inconsistency in the rules for the use of signs. Since such inconsistency concerns the very 'sign-game', it must occur in all applications of the game. And since the rules in question directly determine the meanings of signs, their inconsistency should result in the total destruction of the respective meanings. Wittgenstein should not be credited with this interpretation of inconsistency in terms of the theory of language-games, although it can also be found in his writings. The idea actually goes back to Frege, and Wittgenstein drew on his works in this regard — through the mediation and under the influence of Waismann — just like in the case of the notion of sign-game and the conception of grammar as a sign-game.

In setting out the idea of formal arithmetic, merely outlined by Heine and Thomae, Frege expressly formulated the need for protecting this sign-game by means of a proof of consistency:

> The assertion that formal arithmetic permits of a completely consistent foundation accordingly lacks proof; on the contrary, its truth is subject to grave doubts. Thomae's contrary opinion rests on the mistaken supposition that the rules given in his second paragraph [cf. Thomae 1898] constitute a complete list and especially on his complete unawareness of the prohibitory rules which each new class of figures necessarily requires. (Geach and Black 1960: 215)

Naturally, it could not be consistency in the traditional, semantic sense. Rather, it would have to be a formalist counterpart adapted to the general concept of sign-game and — in the present case — to the properties of the sign-game corresponding to arithmetic. According to Frege — and to no one before him, since the issue had not been discussed in detail by the older formalists — *inconsistency* could only apply to *the rules for the use of signs*. For, granted that signs are nothing more than mnemonic material employed to present the rules of their use, and that the meaning of signs has been reduced to these rules, the traditional inconsistency between formulae or between their meanings must be reassigned to the rules themselves. It can only consist in a situation in which some rules contradict others — in particular when rules defining the use of a specific class of signs contradict other rules defining their use, e.g. general rules concerning numeral signs. Such a contradiction, in turn, can be revealed in the game only when some operations cannot be performed on certain figures or when an attempt at carrying them out fails to bring definite results which could be used in playing the relevant game. Such a 'crash' or self-destruction of a game occurs in the case of failure to impose suitable limitations on substitutions involving the sign 0 ('prohibitory rules' — Frege 1903: § 114, Geach and Black 1960: 210).

Yet how can we characterize such inconsistency in the game rules in wholly general terms? For Wittgenstein, the paradigm case of incoherence and related ambiguities were internally inconsistent formal systems and the well-known antinomies, such as Russell's paradox and the liar paradox. Accordingly, he believed that the common property of incoherent sign-games consisted in the fact that they always involve a 'configuration' in which "I don't know what I'm supposed to do" (Wittgenstein 1964: 319). In the "true/false game" (1964: 321), e.g. in scientific theories and logical and mathematical calculi analysed by Wittgenstein — and generally: in games that involve winning and losing — the inconsistency in the game rules would occur in configurations which (1) are neither winning or losing and (2) are not the starting point for the next move in the game. Inconsistency of the rules within this kind of game could also appear in configurations (e.g. sentences) or positions in the game that can only lead to configurations which are neither winning nor losing. Finally, like in the case of the above-mentioned antinomies, inconsistency may arise in a configuration which is both winning and losing — if it is true, it is false, and vice versa.

Naturally, the distinctive characteristic of the ambiguity determined by the inconsistency in game rules, as shown by Frege in his critique of the older formalism, is that it leads to the total destruction of meanings determined by the rules in question. Such a destruction takes place, as demonstrated in § 117 of *Grundgesetze* (Geach and Black 1960: 212—213), e.g. when in introducing a new sign (a figure in the game) we neglect substitution-prohibitions governing its use. It turns out that the sign-game of arithmetic outlined by formalists leads to a contradiction. For, in accordance with the rules laid down by Thomae, which leave out prohibitory rules for substitutions involving the sign 0, we could — from the configuration "$(3\times 0) = 0$" and the configuration "$(3 \times 0) : 0 = 3$" (obtained from the second law mentioned in the footnote[8] by substituting numeral signs for letters) — derive, by substitution, the configuration "$0 : 0 = 3$," and, by analogy, "$0 : 0 = 4$," which yields "$3 = 4$."

This leads, as shown by the 'proof' of the formula "$3 = 4$" (and we can prove, by the same token, any numerical equalities), to the destruction or extinction of meanings of numeral signs.

In this context, it is quite surprising that despite Waismann's numerous suggestions and arguments, drawn from *Grundgesetze*, Wittgenstein had long refused to accept this fact. This insistence can be explained, to some extent, by his purely operational or pragmatic treatment of the sign-game rules as rules for operations. For, in such a case, inconsistency in the rules results not so much in blurriness of

[8]Rules for numeral signs introduced by Thomae (1898: 1) include classical associative and commutative property for multiplication and subtraction enriched by the rule: $(a' \times a) : a = a$ and $(a' + a) - a = a$. Frege showed that they do not constitute a complete list of rules for arithmetic signs. Most importantly, there is no rule customarily associated with inequality ($\neq$) — or, more generally, negation — and, of course, no prohibition against substitution involving the sign 0.

meanings of particular signs as in unfeasibility of sign operations defined by these rules, that is, in annihilation of meanings of the rules themselves:

What is a rule? If, e.g., I say 'Do this and don't do this', the other doesn't know what he is meant to do; that is, we don't allow a contradiction to count as a rule. We just don't call a contradiction a rule — or more simply the grammar of the word 'rule' is such that a contradiction isn't designated as a rule. Now if a contradiction occurs among my rules, I could say: these aren't rules in the sense I normally speak of rules. (Wittgenstein 1964: 344—345)

The specific type of contradiction between rules, or rather the type of its manifestation, depends, of course, on the properties of the language-game under investigation. The only thing we can generally say about this kind of ambiguity is that in the case of contradiction among the rules we always arrive at a configuration, position, or situation in which the rules fail to determine what we should do, think, or accept next — a situation in which "I don't know what I'm supposed to do" (Wittgenstein 1964: 319).

But can we conceive of such ambiguities as ambiguities of linguistic signs, or should we assume instead that we are dealing not so much with ambiguity of meanings as with nonsensical configurations of signs such as "$2 : 0$," "$3 = 4$," etc.? The former solution seems more plausible, for two reasons: (1) such 'deadlocks' of language can be recognized *a priori*, which suggests that it is a matter of grammar alone, but also that we *are* indeed dealing with a grammar and so with a specific *language*. (2) If we were to assume that in the case of configurations such as "$(3 \times 0) : 0 = 3$," "$2 = 5$," there are no meanings or linguistic expressions at all, then we would be forced to admit that the differences between all these cases, and in particular between various antinomies, are semantically irrelevant. We would have to assume that the liar paradox and the expression on which it rests — "I always lie" — as well as Russell's paradox with the corresponding formula "the set of all sets which are not their own members" mean exactly the same thing, namely — nothing; yet the liar paradox can be constructed just by means of the concept of sentence (not counting the universal quantifier and the propositional negation), while Russell's paradox additionally requires, at least, the concept of set and the concept of membership in a set.[9]

The liar paradox shows that the concept of sentence cannot be used without limitation in its original, naïve sense; thus it is the naïve concept of sentence that

[9] To be more precise, the original formulation of Russell's paradox, in the notation of *Grundgesetze*, involves the generalized primitive relation ($\cap$), for which Frege had established the equivalence: $x \cap \acute{\varepsilon}F(\varepsilon) \leftrightarrow \ F(x)$. By means of this correlation, Frege accounted for the fact that x falls under a concept (F) and thereby belongs to the extension of that concept, i.e. is a member of the corresponding set.

calls for additional rules of use and, thereby, grammatical regulations. On the other hand, solutions to Russell's paradox — even the one proposed by Frege and known as 'Frege's way out', as well as the one offered by Russell's type theory and by set theory — do not involve any semantic modification of the general concept of sentence. Frege's and Russell's solutions alter the grammatical rules determining the use of concepts (or, rather, propositional functions), while the set-theoretic solution modifies the rules for the use (and thus the meaning) of the sign of set membership and the concept of set. What these antinomies show, therefore, is nothing but the ambiguity of two 'naïve' notions — of set and of sentence.[10] Accordingly, a more thorough going examination of the impact of semantic antinomies, especially of Russell's paradox, with regard to semantic changes they triggered, could contribute to explaining an interesting fact in the history of logic and set theory, namely, the separation of the research concerning the content of concepts, or simply the 'science of concepts', from the research concerning the extension of concepts; the former being conducted in the field of logic as the predicate calculus and the latter becoming the domain of, usually axiomatic, set theory.

3.2. Language-games and their accumulations

3.2.1. The 'closed system' paradigm

One might think that the assumption about the meaning of signs, made by Wittgenstein during the formative period of his second philosophy, that such meaning presupposes a system of rules of use (in a sign-game), and that the rules, in turn, must not be contradictory, will effectively make the ambiguity of *signs* disappear from our sight. Naturally, in a coherent, and thereby 'closed', system of rules, no ambiguity is possible. In such a game, e.g. in a consistent calculus, in a system of sentences or formulae, we cannot even formulate a question which would

[10] It would prove useful to investigate more fully the history of Russell's paradox with respect to the semantic changes to which it gave rise. After all, its original formulation indicates that it reveals inconsistency in the rules of use not only in the case of the concept of set, but, above all, in the case of the concept of concept. It was precisely the concept of concept that, according to Frege and Russell, turned out to be unclear, or at least not sufficiently precise. More specific grammatical regulations were needed to define the relation between the content and the extension of a concept. For this reason, Frege believed that it was his Basic Law V, the so-called abstraction principle, $\acute{\varepsilon}F(\varepsilon) = \acute{\alpha}G(\alpha) \leftrightarrow \ \forall x\ (F(x) \leftrightarrow \ G(x))$, that was responsible for the antinomy recognized by Russell. 'Frege's way out' amounted to a modification of the use of propositional functions, including second-order concepts — together with the concept of concept. In fact, both of Russell's type theories (simple and ramified), as well as many other solutions to Russell's paradox, rest on analogous grammatical regulations, that is, on a modification of the rules of use.

lack an unambiguous answer within the system. Such a system is free not only of semantically unclear signs but also — in particular — of undecidable problems or issues, with which we 'don't know what we're supposed to do', as is the case with antinomies:

A mathematical system, e.g. the system of ordinary multiplication, is completely closed. I can look for something only *within* a given system, not *for* the system. What does 242×897 yield? This is a question within a system. There are indefinitely many such questions and answers. I can look for a certain answer only because there is a method of finding it. Algebra (calculation with letters) is also such a closed system, and the same applies to trigonometry as it is taught at school. I can ask, e.g., Is $\sin^2 x = \tan^2 x$? But I cannot ask, Is $\sin x = x — x/3! + x/5! — \ldots$? This is not for the reason that elementary trigonometry is somehow incomplete. (Waismann 1979: 35; in conversation with Waismann and Schlick, 19 December 1929)

The second question cannot even be formulated in terms of elementary trigonometry. Unsurprisingly, therefore, it cannot be answered in such terms. From the viewpoint of the system of elementary trigonometry, we are not in a position to build the formula contained in this question, and so we are unable to raise the corresponding problem. This is the case for the well-known problem of angle trisection. In elementary geometry, i.e. only in terms of compass and straightedge constructions, the problem cannot be framed. Its formulation is possible only within a much richer system, where compass and straightedge constructions can be described, or just expressed, algebraically. In the polemic against Weyl,[11] Wittgenstein generalized this grammatical fact by giving it a logical character — the character of a general law of the 'critical grammar' or the 'grammar of meaning'.[12]

So far as the issue of linguistic ambiguities is concerned, the most important and striking aspect of all these examples is that we are not talking about a *unique* system of rules of sign use — not only in the case of everyday language

[11]See Waismann 1979, 36—37, where Wittgenstein takes issue with (Weyl 1927b: esp. 20—24).

[12]According to Wittgenstein, "Weyl puts the problem of decidability in the following way. Can every relevant [*einschlägig*] question be decided by means of logical inference? The problem must not be put in this way. Everything depends on the word 'relevant' [*einschlägig*]. For Weyl, a statement is relevant when it is constructed from certain basic formulae with the help of seven principles of combination [*Kombinationsprinzipien*] (among which are 'all' and 'there is'). This is where the mistake lies. A statement is relevant if it belongs to *a certain system*. It is in this sense that it has been maintained that every relevant question is decidable" (Waismann 1979: 37, my emphasis).

(*Alltagssprache*) or natural language but also in the case of mathematical signs. Even the most basic mathematical 'calculi', such as trigonometry, arithmetic, etc., consist of even simpler, 'closed' systems or at least contain parts corresponding to such systems. This applies above all to the second trigonometrical expression mentioned above. Wittgenstein observes that with it:

we have in fact moved on [from elementary trigonometry] to a new system that does not contain the old one but contains a part with exactly the same structure as the old system. (Waismann 1979: 35—36)

Another interesting example concerns numbers:

the natural numbers are not identical with the positive integers, as though one could speak of *plus two soldiers* in the same way that one speaks of two soldiers; no, we are here confronted with something entirely new. It is similar when we take the step from elementary trigonometric functions to analytic functions defined through progressions. (Waismann 1979: 36)

The above examples, albeit simple, played a crucial historical role. The first and arguably the most clear formulation of one of the key notions of Wittgenstein's second philosophy, the notion of 'family resemblance', used to explain such unclear concepts as the concept of calculus, number, proposition, language-game, etc., and is directly linked to his analyses of various kinds of number and their relation to the general concept of number.

These early analyses have several advantages, which cannot be overlooked, not only in historical discussions of the development of Wittgenstein's views but also in systematic inquiries into language and its ambiguities. (1) In the interpretations of Wittgenstein's second philosophy and in its applications to linguistics, it is customary to associate each ambiguity with 'family resemblances' and thus to clarify the unclearness of the above-mentioned concepts by means of even more unclear ones. For, in contrast to the (general) notion of family resemblance, the family resemblances between cardinal, natural, rational, real, and complex numbers are exceptionally simple, clear, and distinct. (2) Although in *Philosophical Investigations* and Wittgenstein's other late writings, family resemblances are closely connected with the concept of language-games, they can nonetheless be described by means of a much simpler notion, namely the notion of a sign-game with unambiguously specified rules. (3) The account of family resemblances in terms of 'pure' sign-games lets us distinguish two markedly different types of *systematic* linguistic ambiguity.

The first type of systematic linguistic ambiguity is associated with the semantically uninterpreted 'grammar of words and formulae', i.e. with the grammar of their use within a semantically uninterpreted sign-game of language.[13] Accordingly, such ambiguities can be classified, besides the inconsistency in the rules of use, as *grammatical* or *grammatically determined* ambiguities. They must be separated from ambiguities directly dependent on the interpretation of sign-games (necessary for employing them to describe things, give orders, carry them out, etc.[14]), which could be described, for this very reason, as *semantic ambiguities in the strict sense.*

3.2.2. (G.1, G.2) Ambiguity of signs and concepts

As suggested by the above examples, there are ambiguities of signs and concepts which do not amount to inconsistency in the rules of use but still deserve the title of purely grammatical or 'grammatically determined'. To use the example of the concept of number, discussed by Wittgenstein in its greatest detail, note that as long as we stay within the framework of one system of rules — within the grammar of natural, rational, or cardinal numbers — there is no room for any ambiguity regarding numeral signs or the concept of number. The concept of number, however, is associated with something much more complicated than any of the above systems or calculi. According to Wittgenstein, the general concept of number turns out to be so complicated mainly because there are more than one closed systems of rules of use (grammars) associated with the word "number" and with numeral signs, so that we are dealing not with one but rather with many systems of rules for the use of numeral signs.

Let us adopt, for instance, as Wittgenstein did, Frege's definition of cardinal numbers (in terms of the notion of propositional function, abstraction, and the concept of one-to-one mapping). Natural numbers — defined by induction — form a system consisting of two groups (N, 1, 0, +, ×). All their properties and all numerical equalities are demonstrated by induction. The same two methods (definitions and complete induction) can be used to prove laws concerning rational numbers, although the latter form a much more complex system, namely (N, 1, 0, +, - , ×, :), whose part has the same structure as natural numbers. By analogy, among real numbers we can detect a simpler system corresponding to rational numbers. Naturally, family resemblances between the latter two systems are not

[13] For a precise definition of these expressions and their application in analyses of ordinary speech and natural language, see esp. Schächter 1973: 11—13 [part 1, chap. 1, §§ 7—9], 28—32 [part 1, chap. 3, § 5].

[14] As for the variety of applications of the same sign-game, cf. esp. § 23 of *Philosophical Investigations* (Wittgenstein 2009: 11-12) and Lorenz 1970: 125—128, where — in reference to the paragraph just mentioned — Kuno Lorenz opposes the dominant tendency, in logic and philosophy of language, to 'confine the linguistic basis' to 'statements' and to consider other speech acts, such as ordering, asking, etc., as subordinate and secondary with regard to statements.

the same as between natural and rational numbers. Rational numbers are fractions (quotients of integers), while real numbers are, according to Cantor,[15] limits of fundamental sequences (*Fundamentalreihen*) of rational numbers, or, according to Dedekind, cuts of rational numbers.

Furthermore, Wittgenstein observes that:

A proof for real numbers is not a continuation of a proof for rational numbers but an entirely different thing. If any real number is given, then such-and-such holds for this number too, not because of an induction but because of the rules that I have laid down when calculating with real numbers. Thus such a formula does not mean that such-and-such holds good for all real numbers, but that if a real number is given, then I interpret this formula in such a way that it means that such-and-such is true of the limiting case, and I prove this on the basis of the rules that have been laid down for calculating with real numbers. (Waismann 1979: 110)

G.1. In spite of substantial differences, both natural and rational 2 + 2, as well as real 2 + 2, equals 4, and, in addition, the cardinal number of the union of two mutually exclusive two-element sets is four. Furthermore, each finite cardinal number can be uniquely assigned a natural number and thus we can map the set of finite cardinal numbers onto the set of natural numbers. Operations of addition and multiplication are commutative and associative for all the above kinds of numbers. With regard to the passage quoted above, it is important to note that it is the rules of the calculus (i.e. grammatical rules) laid down for real numbers that accumulate the majority of grammatical similarities between different types of numbers — mainly because these rules are drawn from the grammar of rational numbers. *This is precisely what we call 'family resemblances'*.

Thus there are similar signs among various kinds of number — not only in the sense of a similar or the same material but also in the sense of affinity between some, though obviously not all, rules of use belonging to different language-games. Thus it should come as no surprise that if we read "2 + 2 = 4" without specifying which calculus-game (*Rechenspiele*) or sign-game (*Zeichenspiele*) we are currently playing,[16] we cannot determine the meanings of the symbols occurring in these formulae or the meaning of the formula itself. After all, such signs and configurations of signs can be used in accordance with various systems of

[15] Despite the critique levelled by Frege against Cantor's definition of real numbers (Frege 1903: §§ 71—76), Wittgenstein adopted it at the expense of Dedekind's definition (Wittgenstein 1978: 288—289 [part 1, chap. 5, § 34]).

[16] This is precisely the manner in which Thomae (1898) treated individual mathematical calculi, and in particular the elementary arithmetic and the arithmetic of complex numbers: he used the terms *Rechenspiele* and *Zeichenspiele* interchangeably with respect to them.

rules. This is why, in themselves, they are systematically ambiguous, and it is a grammatically determined ambiguity.

At the same time, this type of ambiguity refers neither to general concepts of particular numbers (cardinal, natural, rational, etc.) nor to the broader concept of number in general (if there is such a thing); rather, it refers to *individual numbers* and their configurations, e.g. to numerical formulae belonging to the above-mentioned mathematical systems. Moreover, these ambiguities vary across numerical systems or even across individual numbers, so that we are actually dealing with a family of grammatically determined ambiguities. It is so because each numerical system (and, to a lesser extent, individual numbers) bears a certain grammatical or 'family' resemblance to the remaining grammatical systems (and analogously — to numbers). After all, an affinity analogous to that between 'natural two' and 'real two' also exists between individual numerical systems *qua* grammars or sign-games, and the corresponding ambiguities are reflected by the variety of definitions of numbers and their interpretations. In general, every ambiguity of this kind — stemming from diverse ways of using a sign, a combination of signs, or a system of signs, in accordance with different systems of rules (or simply grammars) — can be thought of as greater or smaller but it will always be a *grammatically determined indeterminacy of interpretation*. Furthermore, as suggested by the example of the concept of number, it is a *systematic* indeterminacy, grammatically determined and belonging, as it were, to the 'normal course' of language.

Schächter describes analogous cases of interpretative indeterminacy or semantic 'multiplicity' of signs of natural language and seeks their source in the 'specialization' of language, analogous to the plurality of grammars associated with numeral signs, and the emergence of expert vocabulary, e.g. connected with shipping, farming, metallurgy, the work system in a factory, etc. (1973: 15—16 [part 1, chap. 2], 24—36 [part 1, chap. 3, §§ 3—6]). The earlier, simpler and 'unspecialized' vocabulary is embedded in the expert lexicon. In other words, the grammatically richer special languages usually have 'parts corresponding to simpler systems' of grammatical distinctions. Signs of everyday language, once incorporated by the specialists into the more complex system of 'expert vocabulary', gain a new interpretation. Does it mean that they have lost the old one — just as the 'natural two', which is no longer supposed to be ordinary 2 but only +2, which in turn, 'by a logical necessity', so to speak, would have to become the equivalence class of all sequences converging to the same limit instead of being a simple number of objects? Not at all! Even the specialists, depending on their purposes, use the old signs or words in the simpler or in the more complex sense. Moreover, due to the accumulation of various systems of expert grammatical distinctions, signs can be used in one of many 'more complex' meanings. Wittgenstein deserves credit not only for recognizing, before Schächter, the above mechanism of linguistic ambiguities, but also for showing that they are not so much deficiencies or imperfections of everyday language (*Umgangssprache*) as, on the contrary, manifestations of grammatical richness of language — and not only natural language at that.

G.1/I. The above-discussed ambiguities of signs stemming from their interpretative possibilities become much more complicated once the family of 'related' grammatical systems, determining these interpretative capabilities, consists not only of consistent systems. For, if at least some of them are inconsistent, then the interpretative indeterminacies arising in a *determinate* way, i.e. for specific interpretations, are coupled with inconsistencies.

G.1/P. It often happens, of course, that we use a language or its part without access to explicitly formulated rules for the use of signs or without being, for some other reason, fully familiar with them. In such a case, we are forced, so to speak, to read the rules off from the very use of signs. Then, apart from all possible grammatical ambiguities, the list of ambiguities of the language in question includes various ambiguities determined pragmatically.

G.2. There are, however, sign-games whose grammar is, as it were, *in statu nascendi* — not for extraneous and subjective reasons, e.g. due to an imperfect understanding or application of the rules of sign usage, but rather for intrinsic reasons. This was the focus of Wittgenstein's later writings, also continually emphasized by Schächter with respect to ordinary language. The resulting semantic instability of signs is determined by the *instability of the rules for their use*, that is to say, it is determined grammatically, similarly to I and G.1. We find perfect examples of such ambiguities both in natural languages — in which every synchronic state, every abstraction from continuous changes of meanings, can only be obtained by means of an arbitrary decision — and in the languages of particular scientific disciplines, including mathematics and logic.

After all, during every crisis of foundations (e.g. the crisis of the foundations of mathematics at the dawn of the 20^{th} century), we problematize the established methods, inference rules, criteria for acceptability of results, admissible experimental methods, types of definition, other 'specialized' foundations, and, in consequence, also the basic concepts, principles, and axioms. As shown by Thiel (1972, 1995: 330—337), two conditions must be met before we can speak of a crisis of foundations or a paradigm shift in a given discipline. First:

certain social groups responsible for its organization (usually the scientists working in a given field, but also public opinion) must reflect on the scientific practice (*Wissenschaftsbetrieb*) of this discipline,[17] voice justified doubts about its results

[17]The term *Wissenschaftsbetrieb* might be rendered as "science factory" or even better as "science-forming enterprise." For we are talking here about the totality of theoretical and material resources employed to achieve scientific knowledge in a given field and the organized groups of scientists exploiting them.

(theoretical sentences and technical instructions) or procedures used to obtain them, and demand changes in this practice. (Thiel 1995: 333)

Besides, a genuine controversy over foundations presupposes alternative proposals for changes of practice within the discipline, proposals leading to new foundations. Hence a controversy about foundations:

arises when influential groups of scientists seek to realize mutually exclusive proposals aimed at overcoming the crisis of the foundations of their discipline. (Thiel 1995: 333)

Each foundational crisis is thus determined, in equal measure, by the state of knowledge and by the 'grammar' of the language of a given discipline — as illustrated by the set-theoretical antinomies underlying the crisis of the foundations of mathematics — and by various pragmatic or sociological factors (no language, even the language of mathematical logic and 'pure' mathematics, is free of them). Nonetheless, once a crisis of foundations or, more generally, a paradigm shift is already present, i.e. once alternative foundational proposals are available — as illustrated by such projects of founding mathematics as logicism, intuitionism, and formalism — it necessarily results in a global destabilization of rules of the language of a given discipline. The crisis of foundations of mathematics in the wake of the 20^{th} century concerned not only its logical basis — especially the account of concepts and the abstraction principle (Hilbert 1922: 162) — but also the list of admissible methods of proof, and, above all, the principle of mathematical induction, criticized by Frege and Russell, as well as the principles rejected by intuitionists (the law of excluded middle and De Morgans's laws for quantifiers).

Thus, unsurprisingly, the destabilization of rules determining the meanings of fundamental notions brought to light the disputable nature, and — more importantly — *ambiguity*, of central mathematical concepts, such as the notion of natural number (thoroughly discussed by Frege in *Grundlagen der Arithmetik*), rational, and real number (e.g. Frege 1903: §§ 67—90, Thomae 1898, 1906, Hilbert 1922), the classical concept of set (criticized and rejected by intuitionists and Wittgenstein), the notions of function, continuum, continuity, etc. Thus the conceptions and theorems of what in the age of crisis, without any deeper historical or methodological reflection ('roughly', so to speak), was called 'classical mathematics' — usually referring to the body of established mathematical concepts, theorems, and methods — became radically and systematically ambiguous. Their meaning started to depend on the choice of critical arguments against the inference rules and the logical foundations of mathematics that were considered plausible, and on the position one took on the issue of founding the 'classical mathematics' — for this is what determined the set of accepted grammatical rules defining the meanings of fundamental mathematical notions.

Accordingly, in the age of crisis, natural numbers could have meant, and did mean, a variety of things. They could have been defined in Frege's and Russell's way, by means of the concept of cardinal number, which in turn was defined with the help of the notion of propositional function and its extension. Yet if one rejected — like Hilbert — 'the logical notion of concept and its extension' as paradoxical, they could have been defined in the framework of Cantor's or Zermelo—Fraenkel set theory. One could also refuse — like Wittgenstein at the time of the *Tractatus* and intuitionists — to accept set theory and, instead, define natural numbers as the exponents of operations[18] or arguments of a fundamental sequence. Finally, they could have been understood in purely formalist terms, as figures in the sign-game of elementary arithmetic. Equally diverse were the possible meanings of rational and real numbers, of sets (including the uncountable ones), of the concept of function, and of many other mathematical notions.

We should bear in mind, however, that the kind of ambiguity discussed here has little to do with polysemy of concepts — rather, it consists in their vagueness or fuzziness. Although the ways of understanding numbers etc. were determined by alternative choices concerning the foundation of mathematics, and so by different grammatical regulations, all alternative interpretations preserved a common 'semantic core' of these concepts, equated with their 'classical' meaning, or rather with their meaning within 'classical mathematics'. Besides, the type of ambiguity in question — *pace* Wittgenstein — cannot be reduced, like in the case of G.1, to the occurrence of the same concepts across a family of related calculi. The crisis of foundations and the accompanying instability of rules is independent of whether the relevant grammar contains similar sign-games, together with signs and formulae affected by this family resemblance. The instability might as well affect a language in which no sign-games interfere or overlap with elements of other games.

In the analysed example, the crisis of foundations and the destabilization of grammatical rules did indeed bring out family resemblances characteristic of the concept of number, and — at least in the case of numbers — resulted in the conjunction of the two types of ambiguity. Consequently, individual natural (and not only natural) numbers became exceptionally convoluted and ambiguous. Due to the alternative attempts at reforming the grammar of numbers (functions, sets, ...), the language of mathematics did not contain any unique concept of natural or real number, even defined in terms of family resemblances. This was not due to interferences or overlaps within the family of related sign-games but rather

[18]This is the definition accepted by Wittgenstein in the *Tractatus*, 6.02—6.03 (1961). Frege had given a very similar definition, using the concept of series, in the third chapter of *Begriffsschrift* (1970). The intuitionist concept of natural number resembles Wittgenstein's as well.

because the basic rules of this whole variety within mathematics became debatable and 'fluctuating'. Of course, there remained numeral signs common to the whole mathematics (and, analogously, signs for functions, sets, relations, etc.) and some universally, or almost universally, accepted rules for their use, constituting the contents of 'classical mathematics'. Still, mathematicians may have, and did have, doubts about their meaning.

3.3. Ambiguity of 'excessively general' concepts

Given such a complicated constellation of sign-games, are we dealing with a *unique general concept*, e.g. a unique general concept of number, which encompasses all figures involved in the overlapping sign-games entangled in the crisis which gives rise to the instability of grammatical rules, or just with one word, which may mean different things depending on which specific regulations of grammatical foundations one stipulates (e.g. which project of founding mathematics one endorses)? It depends on what we call concepts, that is, on the notion of concept we adopt, and on whether we truly have one concept adequately representing entities as diverse as e.g. cardinal, rational, . . . , complex numbers in their multifarious interpretations; interpretations which in turn depend on decisions concerning the foundations of mathematics. If the proposed systems of foundations are mutually exclusive, as is the case with intuitionism and formalism (or, previously, logicism), then it is impossible, of course, to arrive at a general and universally accepted concept of number. An analysis of the phenomena accompanying crises in other branches of science and the history of particular natural languages might offer numerous analogous examples.

Still, once we overcome a crisis and the associated instability of rules, are we in a position to find a common concept for the figures and configurations entangled in the grammars of related sign-games? To put it another way: do the languages assembled from related sign-games always contain concepts that can adequately capture all figures caught up in these families of sign-games? One thing is certain — in its own right, the notion of family resemblance does not secure a general concept capable of capturing all 'figures', 'configurations', or 'positions' constituting the related sign-games. As shown by Wittgenstein in *Remarks on the Foundations of Mathematics* (1978), we have no *general* concept of number capable of comprising numbers of all known systems of numbers — even independently of the crisis of foundations. Actually, we only have access to a *collective* concept of the above-discussed family of various sign-games or systems of numbers and to the concepts of grammatical similarities between them.

Nevertheless, we use the concept of number as though we not only had a general concept of cardinal number, a general concept of natural number, etc., but also, on top of that, a general notion encompassing cardinal, natural, rational, and other numbers. The same applies to concepts such as game, calculus, language, proposition, expectation (Wittgenstein 1958: 20—22), knowledge (Wittgenstein 1958: 22—24), and the like. All of them, similarly to the concept of number, are systematically ambiguous, and this ambiguity is determined by the grammar of the sign-games in which one deploys them. According to Wittgenstein (1958: 17—20), these ambiguities are due to our 'craving for generality' — the tendency to equip words (and especially names) with unlimited, precise generality. In this connection, we can distinguish two principal kinds of ambiguity.

G.3.1. We usually use words: proposition, number, game, knowledge, etc. *in reference to a limited domain* of linguistic systems or language-games which form such-and-such 'our languages'. Then these words play the role of chapters in the handbook of grammar for these languages. In order to learn the meanings of such words and dispel all doubts, it is enough — yet it is no small task — to describe their various uses. But if, in addition, we wish to speak of *a unique concept* encompassing all these grammatical types and subtypes, then, more often than not, it will not be a *general* notion in the ordinary sense, such as the general concept of natural number, but rather a mere *collective* concept; this is due to the differences between systems of rules for the use of different types of proposition, number, game, types of 'knowledge', etc.

Schächter was reluctant to think of such words as ambiguous or vague (1973: 12—13 [part 1, chap. 1, § 8]). Rather, he was inclined to treat them as examples of polysemy or 'multiplicity of meaning'. We use the same sign material, the same word, e.g. "constitution," analogically within different systems of rules of use: "natural condition of body or character," "basic guiding principles of government," "material make-up of a substance," etc. (Schächter 1973: 13). In the languages of particular branches of science, it often happens that a word borrowed from ordinary language is given an entirely new meaning. Schächter's examples include signs or names in the field of physics, such as "work," "force," energy" (1973: 13—14). What these signs, as used in ordinary language and in physics, have in common is mainly — or, according to Schächter, exclusively — their material.

The rules for their use (their grammar, or — more precisely — the grammar of their use) in everyday language and in the language of physics are quite different. There are undoubtedly some grammatical similarities between them, for instance, in both systems of rules force, energy, and work have quantitative characterizations. But is this enough to form one general concept of force capturing all cases of 'physical' and 'everyday' force? Indeed, is the everyday notion of force a single *general* concept, or just a chapter in the grammar of 'our' everyday language? Should we consider the everyday notion of force, or some version of it, as the *prototype*, as a 'more genuine' force, while all other versions as more or less similar to that prototype force? Would such an account of a 'general' notion of force adequately capture the actual meaning, or rather meanings, associated with this word?

G.3.2. Still, even in situations such as the ones described by Schächter with regard towards "force," "work," "energy," we are talking about a kind of grammatical ambiguity rather than mere polysemy. Usually, however, these are ambiguities introduced to semantic analyses by logical and linguistic terminology and only 'secondarily' lent to the meanings reconstructed by its means. Ambiguities arising at the interface between language and its descriptions spring from the fact that the analysis of language does not recognize, as a rule, any 'natural' units other than sign, sentence (proposition), and language. The sentence and the sign are defined within the framework of the language to which they belong, which boils down to the dilemma: either within the *system* of the natural language or within the *calculus* to which they belong.[19]

This also accounts for the *terra incognita* between formal languages, or simply calculi, and natural languages — populated by languages which, like the language of mathematics, physics, sociology, philosophy, or the language of everyday life (*Alltagsprache*), are neither artificial (are not mere calculi) nor natural, and, more importantly, fail to form a *unique* calculus or system. Languages of this kind, as shown by Wittgenstein and Schächter, actually consist of numerous, more or less complex 'sign-games', closed 'systems of grammatical distinctions' constituting their exceptionally intricate and radically heterogeneous grammar of meaning. This non-systemic character — as emphasized in particular by Schächter — also marks German, English, Polish, Chinese, and any other *language of everyday life* (*Alltagsprache* or *Umgangsprache*), in which various signs are used in accordance with distinct systems of rules, in part derived from mathematics, physics, building, shipping, chemistry, computer science, law, economy, and so on, and so forth. Thus it should come as no surprise that in such a mixed bag of grammars many signs lack a homogeneous system of rules of use or a uniquely determined set of designata.[20] Since such ambiguities — whose instances are, among other things,

[19]Clearly, there are numerous considerable differences between the logical conception of language as a calculus and the linguistic, especially mentalist, notion of 'natural' language. Yet they are fairly similar in at least one semantically relevant respect: it is always the linguistic *system* that serves as the target of questions about the meaning of signs and grammatical forms. For generativists, as well as for logicians, a given language is simply a *unique* calculus whose signs and their meanings are defined — for 'obvious' reasons, associated by Chomsky with the 'creative character' of language — by means of ('surface' and 'deep') recursive rules. Also in both European structuralist schools (Prague and Copenhague), meaning of a sign is characterized, by and large, in terms of its position (synchronic value) within the *whole system* of language. Jakobson, seeking to unify the semantic heterogeneity of signs of natural language, defines their meanings as semantic invariants across various ways of use (Jakobson 1971: 225). The same idea of the *systemic nature* of language also underlies the theory of prototypes.

[20]An attempt to find unity and systemic nature in the grammatical diversity of everyday language (*Umgangsprache*) lead Schächter to an interesting conclusion applicable also to the notion of 'natural language' (which is a counterpart of the 'language

'prototypal categories' in natural languages — are only secondarily introduced to the reconstructed language via its various theories, they should be set down to the metalanguage(s) rather than to the reconstructed language itself and labelled *metalinguistically determined ambiguities*.

G.3.3. Another type of grammatically determined ambiguity occurs when, as pointed out by Wittgenstein, one wants to use words such as number, language, game, proposition, wish, etc., not only as collective concepts but also as general ones. One is then compelled to *propose*, on the basis of the known and clear family resemblances, a *general* concept which could be adequately applied (without confining it to the above-mentioned 'chapters in the handbook of grammar') to countless other (distinct from what is already known) examples of numbers, games, propositions, wishes, etc. Clearly, (1) a concept of this kind must be a more or less arbitrary semantic stipulation — a new rule for the use of a definite, already existing sign; (2) if we subsequently wish to transform such a definition into a general concept, we must adapt it to new 'grammatical systems', to as yet unknown (and not described in any grammar) numbers, games, propositions, languages, and so on. Hence, every such concept is doomed to various modifications.

It is exactly for this reason that Wittgenstein said that such concepts *dissolve* (1974: 119). Outside of the already known fragments of various grammars, their

of everyday life', at least in the domain of contemporary languages such as German, Polish, Chinese, or any other): "only certain words and their combinations fall under the imprecise description of őeveryday languageŕ, and these are common to all special languages of sailors, factory workers, farmers, engineers and so on" (1973: 15 [part 1, chap. 2]).

Among such signs, Schächter only listed some propositional conjunctions, such as "and" and "not," and words denoting activities that are 'independent' of a particular profession or a special language, such as "go," "eat," "table," "left." It is doubtful, however — and such doubts predominate in Schächter's work — whether there really are words whose meaning is utterly independent of various human activities or more or less specialized languages. It is even more doubtful whether one could piece together all these words and isolated, partial, so to speak, rules of their use, so as to fashion any of the contemporary 'natural' languages. Still, there is no denying — and Wittgenstein and Schächter did not deny it either — that natural or everyday language indeed forms a certain whole marked by a kind of unity.

Yet it is also hard to imagine that this would be a whole or a unity other than the one found by Wittgenstein in mathematics of the age of the crisis of foundations and by Schächter in the everyday language of his time. Both languages form, to an equal degree, a 'network' of criss-crossing sign-games tied together by numerous family resemblances. Furthermore, this network is constantly amenable to changes — both as a whole and in each of its subsystems. Wittgenstein described its general properties *inter alia* by means of the metaphor of suburbs: "Our language can be regarded as an ancient city: a maze of little streets and squares, of old and new houses, of houses with extensions from various periods, and all this surrounded by a multitude of new suburbs with straight and regular streets and uniform houses" (Wittgenstein 2009: 8 [§ 18]).

content and extension is in constant change, and the direction of that change cannot be predicted or controlled *a priori.* In *Philosophical Grammar* (1974: 114—121 [§§ 71—76]), Wittgenstein regarded this kind of ambiguity as typical of most, if not all, grammatical notions supposed to describe — with unrestricted generality — certain grammatical items such as number, proposition, word, game, language, rule, calculus, representation, etc. Characteristically, with respect to the known types of games etc., they can be applied as collective concepts, encompassing certain grammatical systems or even general concepts of some grammatical items (such as a number or a proposition) fashioned within such systems. But if we try to transform them into general concepts then they necessarily 'dissolve', just as the general concept of game does:

For us games are *the* games of which we have heard, the games we can list, and perhaps some others newly devised by analogy; and if someone wrote a book on games, he wouldn't really need to use the word "game" in the title of the book, he could use as a title a list of the names of the individual games. If he's asked "but what's *common* to all these things that makes you collect them together?" he might say: I can't give it straight off — but surely you may see many analogies. Anyway the question seems to me idle [*müßig*], because proceeding by analogy, I can also come by imperceptible steps to things that no one in ordinary life would any longer call "games." [. . .] The case is the same with the concepts 'rule', 'proposition', 'language', etc. (Wittgenstein 1974: 116—117 [§73])

This is exactly how — according to Wittgenstein — all the above-mentioned *general* concepts (of number, proposition, game, calculus) 'dissolve'.

3.4. Indeterminacies

3.4.1 (G.4.1) Indeterminate conceptual characterizations

Linguistic ambiguities of this kind, well-known in contemporary linguistics, can be illustrated by almost any name or concept of natural language. Schächter uses, *inter alia*, the example of "to obey an order" ("x obeys an order"). Can we say that trained animals follow orders? Yes and no. The rules for the use of the word "order," like in the case of many other words of everyday language, allow us to distinguish not two but three areas of use:

to the first belong all those cases where usage admits a sign; to the second, all those where it excludes a sign; and to the third, all those where it allows no decision. [. . .] Suppose someone ask us: can we say that animals trained to approach on hearing a certain sign of a bell obey this sign? Here we must note the following: (i) When applying this word to man, the question simply does not arise, usage is unambiguous. (ii) For objects like table and chair and so on nobody (except animists) will speak of obeying. In between we have the reactions of animals and plants, which more or less resemble either man or object. To the above question we would

reply: we are free to call this behaviour 'obeying' or not. (Poets extend the first area at the expense of the second, the sleeping apple, the laughing sun, the merry wind and so on.). (Schächter 1973: 12 [part 1, chap. 1, § 8])

Applying this expression to humans is uncontroversial; the same is true of not applying it to trunks or stones. However, when it comes to donkeys, Pavlov's dogs, and other animals, we are no longer certain if they too *are able to* 'obey orders'. The rules of language do not specify whether their behaviour can be described in these words. Arguably, Schächter is on the right track when he claims that "in that case a question as to membership of a borderline case [in the extension of a concept] is misconceived. For here language has by convention renounced the question" (1973: 14 [part 1, chap. 1, § 8]). The justification of this view also seems plausible:

or rather, its concepts are as though defined in a way that precludes such questions from arising: if someone asks them, he must have defined the concepts differently. (Schächter 1973: 14)

We cannot, however, avoid the question — which is perhaps partly diachronic or even genetic in nature — about the origin of such ambiguities or misconceptions. Do they always stem from the fundamental indeterminacy of the rules for the use of signs in natural language, where they are 'tacitly' established in the course of using the signs ('as we go along') and so are subject to the elusive influence of non-grammatical factors? If this really were the only source of the ambiguities in question, we would be talking about a change in meaning, a topic for pragmatics or simply for historical linguistics. Schächter himself was often inclined to understand the indeterminacy of the scope of applicability in this way.

It must not be overlooked that ambiguities of this kind take place in a synchronic state of language: that they are part, as it were, of its 'normal course'. After all, we often apply certain concepts to areas for which, according to the traditional grammar and the ordinary use, they were not meant. This is the case with "obeying orders," as well as with the majority of Lakoff's and Johnson's concepts marked by metaphorical structure. In fact, every concept can be used outside of its originally established area of application, and — as pointed out by Lakoff and Johnson —it is a perfectly normal way of using concepts, not only in natural languages but also in the languages of particular sciences and other specialized languages.

3.4.2. (G.4.2) Conceptual inaccuracies

There is a related case of indeterminacy, which may be dubbed *conceptually determined*, or just conceptual, *inaccuracy*. Suitable examples include expressions like "until Friday" and "in the daytime." Does the expression "in the daytime" also refer to the last minute of the day or maybe even to the first minute of dusk? In a variety of conceptually defined orders and series, we encounter doubtful elements or even whole transitory areas and unspecified boundaries between them. Here again, language, with all its fixed conventions, gives us carte blanche to subsume a given 'borderline case' under a given concept. It is only a matter of making the rules of use more precise, wherever 'language has renounced' precision because, so far, it has not been important.

Are these ambiguities also connected with the fact that 'our language' (natural language) does not conform to the law of excluded middle, or, more precisely, to the *postulate* of sharply bounded concepts, explicitly put forward by Frege but accepted throughout the history of classical logic? Are these ambiguities characteristic of the 'logic of natural language'? Do they apply to the concepts of natural language alone or to scientific concepts as well? Perhaps they arise because concepts within 'natural language' are for the most part devoid of 'logical structure'; rather, they are defined in terms of 'paradigm cases' — prototypes — and roughly specified similarity to them, so that they can neither have unambiguous, clear contents nor — *a fortiori* — a clear-cut, sharp extension?

Presumably, in some cases the prototypal character of 'natural categories' may account for such phenomena. We should note, however, that — as pointed out by Wittgenstein in his analysis of the visual field (*Gesichtsfeld*) — the ambiguities under discussion rest on two assumptions. They are possible only if (1) we are dealing with two different scales — otherwise the request for a more precise specification could not even be formulated; in fact, neither of these scales need be more precise than the other (as opposed to the visual and geometrical space described by Wittgenstein): in order to enable requests for a more precise specification, it is enough that the scales are different and that (2) they concern the same 'conceptual characterization' (the same 'logical coordinate' or 'parameter') of a given class of objects. Of course, ambiguities usually occur when the same property of objects can be described both by means of a finer and by means of a coarser scale. This happens, for instance, when one attempts to translate ordinary descriptions of time, place, colour, sound, etc. into mathematical and physical continuous scales. Accordingly, it must be emphasized that ambiguities of this type are possible only if 'our language' not only contains distinct conceptual systems but also allows us to apply them to the same 'conceptual characterization' (the same property) of a given class of objects.

4. Semantic ambiguities (S.1, S.2, S.3)

All the above kinds of ambiguity are determined grammatically and may occur in pure sign-games as well as in semantically interpreted games, that is, in languages.[21] In *Philosophical Remarks*, Wittgenstein (1964:73—81 [§§ 38—48]) made use of the notion, or rather a metaphor, of a proposition as a yardstick or a ruler laid against reality, and with the help of the examples of the unit length, colour samples, etc., worked out one of the key semantic notions — the concept of means of representation (*Mittel der Darstellung*, Wittgenstein 2009: 25 [§50]). It then became evident to him that:

It's easy to understand that a ruler is and must be in the same space as the object measured by it. But in what sense are words in the same space as an object whose length is described in words, or, in the same space as a colour, etc.? It sounds absurd. [. . .] The unit length is part of the symbolism. It belongs to the method of projection. Its length is arbitrary, but it is what contains the specifically spatial element. And so if I call a length '3', the 3 signifies via the unit length presupposed in the symbolism. You can also apply these remarks to time. (Wittgenstein 1964: 78—79 [§ 45])

By incorporating such elements of reality, sign-games become games with reality, *language*-games in the strict sense — forms of life (Wittgenstein 2009: 11 [§ 23]). Once we deprive them of this embedding in reality, say, in order to explore them in purely formal terms, they turn into pure sign-games, which might as well be associated with different 'units' (unit lengths such as inches, feet, yards, etc., colour samples, sound intervals, weights, and so on). A 'pure grammar', that is, a sign-game, remains the same under all these interpretations. Yet its symbols and formulae signify something else in each case, depending on the means of representation to which they have been linked. Clearly, all language-games, including games with reality, that are based on the same sign-game are isomorphic, or at least homomorphic. This shows that, on the one hand, pure grammar determines, to a large extent, meanings and the whole semantics of language, yet, on the other, leaves unlimited space for possible semantic interpretations; that space is the subject matter of model theory.[22] At the same time, it is the

[21] It is precisely the interpretation, or the 'application' of a sign-game, that Wittgenstein saw as the key difference between sign-games and languages (Waismann 1979: 104).

[22] For these reasons, model theory should not be regarded as a descriptive semantics of language (in any case, that was not Tarski's intention). His concept of truth and model theory, like the 'method of models' before, is not a semasiology in Marty's sense but, above all, an instrument or method for proving consistency of deductive systems, subject to rigid restrictions regarding its applicability. For these among other reasons,

space of various possible ambiguities connected with interpretation, that is— in contradistinction to the ones discussed so far — *semantically* determined ambiguities.

Yet even a preliminary analysis of these phenomena requires additional elucidations. As we have seen, interpretations of a sign-game (like the whole grammar) are based not on sentences (propositions) but on rules. The choice of a unit length etc. — whether it has been made 'consciously' or 'tacitly' — amounts to a *stipulation* concerning meaning, and in a special case — to a kind of habit. In any case, however, it is not a sentence (proposition) which might be verified or falsified. The same is true of the connection between the means of representation and a particular sign of language together with the rules of its use. Accordingly, the process of interpreting a sign-game can be viewed both from a grammatical and a semantic viewpoint.

S.1. Separation of a pure grammar — a sign-game — from its applications lets us distinguish, at least within sign-games with reality, a new kind of ambiguity. It stems from the diversity of means of representation and their associations with distinct systems of rules of use — with different grammars. It happens both that (1) a linguistic community uses the same fragment of reality (means of representation) according to similar but different rules and that (2) the same grammar is associated with different means of representation. Good examples of the latter are various medieval ells and ounces as well as the tone *a'* in Bach's times.

S.2. The reverse holds if the means of representation are unique — if, say, it is a single object or situation — but the rules according to which we want to use them remain unclear, ambiguous, or even indeterminate. For this leads to blurriness of the very means of representation. It becomes unrecognizable. We cannot tell with which syntax we should associate it, according to which rules we should use it, in short — what it is supposed to mean. Naturally, in such a situation, ambiguity also affects the sentences presupposing that symbol or at least appealing to it. It marks all ostensive definitions and all predicates or 'predicators' (Lorenzen 1987: 25f) introduced by means of exemplars. Apart from ostensive definitions, Wittgenstein's favourite example of this kind of ambiguity was a pointing gesture or — in other words — the meaning of words such as "this," "here," "there."

S.3. Finally, semantically determined ambiguities should include ambiguities resulting from translation. Although translation is never a sufficient means of radical semantic interpretation, it often serves to clarify meanings and always yields an interpretation of one symbolism in another. We must bear in mind,

Wittgenstein rejected as insufficient all semantics based on translation, such as the 'method of models', and — in order to avoid 'the vicious circle of analytic philosophy' (Wittgenstein 1958, Lorenzen 1968) — he kept calling for 'radical interpretation'.

however, that from the viewpoint of Wittgentein's and Schächter's philosophical grammar, we can speak of a *translation* from one language into another language, from one symbolism, system, or calculus into another symbolism, system, or calculus (possibly belonging to the same language as one of its subsystems) only if both languages or systems are marked by a different grammar of meaning, different means of representation, or by both. Otherwise, we are dealing not so much with translation as with transcription.

Of course, translation thus understood, as a tool for explicating meanings, is ambiguous by nature, if it is possible at all. After all, nothing can guarantee that a system of rules for using a given sign material should be expressible in terms of a separate system of rules, or that one system of rules for deploying certain means of representation should be unambiguously and accurately represented by another system of rules for using different means of representation. As shown by Wittgenstein in *Remarks on the Foundations of Mathematics* with regard to 'classical mathematics' and its logicist explication, such translation is indeed possible in some cases. Even so, in order to determine its accuracy and thereby the lack of *ambiguity*, one must know both languages in the first place. Otherwise, even an exact and explicit translation is radically ambiguous— even if it serves as a means for communicating meanings of the translated language to a person who is as yet unfamiliar with that language. It is also unclear whether ambiguities, inaccuracies, and inadequacies connected with translation should be considered as semantic, grammatical, or pragmatic in character — or perhaps they involve all three kinds of determination.

5. Pragmatic ambiguities (P.1, P.2)

P. The issue of a unique assignment of meanings is further complicated if — apart from the rules of use and means of representation — we take into account the very sign material and the process of communication. The list of ambiguities is then extended to include all problems surrounding the connection of such-and-such signifiers with the rules of use and the means of representation. The resulting ambiguities apply not so much to the meaning of signs. i.e. the rules of use as such, as to reading these signs off from their use in the process of communication. Accordingly, they may arise even when the rules for the use of signs of a given language are, in their own right, coherent and explicit, and — in addition — the means of representation for a given sign-game are also uniquely defined. All signs — utterances, written marks, or other communicational activities — even if their grammatical and semantic foundations do not raise doubts, always leave open the issue of their interpretation by the receiver. After all, a given utterance, example, activity, etc., can be in accordance with a greater number of rules, and with each of them to no lesser degree than with the rest of them.[23]Accordingly, each sign

[23]The following example, adduced by Wittgenstein in *Blue and Brown Books*, per-

can be placed in different grammatical and semantic environments or contexts and thus understood in different ways.

P.1. First, we should discuss the original haziness of the rules of language — as opposed to the grammatically determined one — connected with the necessity of reading the rules off from the use of signs. It characterizes any process of seeking understanding (*Verständigungsbildung*). Of course, we might distinguish its various types, depending on whether meanings are read off in isolation from any linguistic system (the situation of a child learning their mother tongue) or whether the receiver decodes the message in the framework of a familiar linguistic system. In *Prolegomena*, Schächter quotes the following example which may throw light on this situation:

when trying to explain to a child that 2 + 2 = 4 by using the words "you have two apples and you are given another two, how many do you have then?", the pupil replied that he had no apples nor had anyone given him two more. (Schächter 1973: 17 [part 1, ch. 2])

Is it a matter of ambiguity of the concept "2," of the concept of addition, or maybe we are just dealing with a stubborn child who refuses to learn the rules for the use of numbers, either by way of reading them off from the use of numeral signs or in any other way? Clearly, the latter can always happen. Such a situation illustrates a borderline case in which we can no longer reasonably speak either of ambiguity or of misunderstanding. The child, or another receiver of a message, just refuses to play the communication game.

P.2. In the above scenario, however, the student is not so obstinate. The teacher eventually manages to provide a rationale for the formula 2 + 2 = 4 and to explain its meaning by invoking a situation from the boy's life:

But he did not succeed until by accident he hit upon a real circumstance in the child's experience by asking him to say how many pairs of shoes he had, and the reply came "one for Sundays and one for weekdays." Now he found it easy to elicit the fact that each pair consisted of two shoes making four in all. (Schächter 1973: 17 [part 1, chap. 2])

Does this didactic success mean that the boy understood the meaning of

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fectly illustrates the general principle of ambiguities connected with reading linguistic rules off from the use and the widely-discussed issue of rule-following: "Someone teaches me to square cardinal numbers; he writes down the row 1 2 3 4, and asks me to square them. [...] Suppose, underneath the first row of numbers, I then write 1, 4, 9, 16. What I wrote is in accordance with the general rule of squaring; but it obviously is also in accordance with any number of other rules; and amongst these it is not more in accordance with one than with another" (Wittgenstein 1958: 13).

natural numbers? Even if he grasped the rules of the sign-game, so that both of them — the student and the teacher — reached an agreement as to the grammar of arithmetic statements, there is as yet no agreement about the application of these formulae. The boy is reluctant to apply numbers and their sums to counterfactual things and situations, so he consistently refuses to assign any meaning to many arithmetic operations carried out 'on paper only'.

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