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FUZZY SETS AS EXTENSIONS OF COMPARATIVE CONCEPTS

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The extension of a name in a given sense (the extension of a concept) is a set of all designata (referents) of this name.[1] Such a description of the meaning of the term "extension of a name" is in need of further explication. In particular, it is necessary to specify the meaning of the word "set."

It is customary to distinguish two meanings of the term "set" — the collective and the distributive sense.

Generally speaking, in the case of a collective set, elements of the set are parts of the set, and, consequently, elements of the elements are also elements of the set (a part of a part of a whole is also a part of this whole). As an example of a collective set, consider the territory of Poland. A formal theory of collective sets is the subject of Stanisław Leśniewski's mereology.

The set of natural numbers is an example of a distributive set. Each element of this set is a natural number. Generally, for distributive sets:

x is an element of the set X

means that

x is an X.

[1]In what follows, the term "name" will be used as an abbreviation for the expression "name in a given sense."

Thus a distributive set is described here as the extension of a name X. Such an understanding of distributive sets is characteristic of the traditional logic. It was employed by the inventor of set theory, Georg Cantor. In set theory, i.e. in the mathematical theory of sets in the distributive sense, the concept of set is a primitive notion. Its meaning is established by a given axiomatic system.

In this article, we will deal with sets in the distributive sense.

The primitive term of axiomatic systems of set theory is $\in$. This symbol denotes the relation of membership in a set (being an element of a set). Each set-theoretic axiomatization, "$x \in X$" is understood in such a way that it is either the case that $x \in X$, i.e. x is an element (member) of X, or it is not the case that $x \in X$, i.e. x fails to be an element of X. Membership in a set is not subject to any gradation. It is not the case that something is an element of a set only to a certain degree — more or less. All objects which are elements of a given set are elements of this set to an equal degree. Sets of this kind, where membership is not amenable to gradation, will be called ordinary or classificatory. The latter terminological proposal is motivated by the fact that all and only extensions of classificatory concepts are sets of this sort (Pawłowski 1977: 109).

This gives rise to a question whether extensions of names are always ordinary sets, that is, whether all designata of a name are elements of that name's extension to an equal degree.

Consider the name *juvenile*. Are we inclined to say of each human being that he is, or is not, juvenile? Intuitively, we allow for gradation. With respect to some people, we are more inclined to say that they are juvenile than we are with respect to others. In applying the name *juvenile* to both of these groups, thereby treating all of them as designata of that name, we must admit that the relation of membership in a set is gradable.

Of course, with a specific purpose in mind, we can 'sharpen' (make more precise) the meaning of the name *juvenile*. In such a case we speak of a precising, or regulatory , definition. In fact, this operation consists in characterizing the extension of a given name as an ordinary set. The name comes to function in the language as a classificatory name semantically associated with the word which it is supposed to sharpen.

In some cases, it is recommended to replace a name "$\mathscr{N}$" with a classificatory name obtained from "$\mathscr{N}$" by means of a regulatory definition. This is desirable, for instance, in the case of legislation. The word *juvenile* becomes a classificatory name after specifying the age range. In the case of other languages, regulating the meaning by characterizing the extension as an ordinary set might be unwelcome: it turns out that in some disciplines, especially in humanities, we use concepts such that any attempt to sharpen their meanings by way of a reduction to classificatory concepts may lead to significant impoverishment of those disciplines (e.g. ceasing to regard certain sentences as true), which would prevent them from playing their cognitive role. Nevertheless, a formally sound and materially adequate specification of extensions is a prerequisite for applying modern-day formal-logic and computer-science tools.

It is a well-known fact that in a number of sciences certain concepts are such that their extensions are not ordinary sets, and their reduction to classificatory concepts is undesirable. This fact constitutes one of the arguments for the claim that formal tools are of limited, if not minor, significance for these sciences. This is not to say that methodologists of humanities renounced any formal description of the structure of non-classificatory concepts. Still, this task is hindered by limiting the concept of set, taken as an extension of a name, to the notion of ordinary set. Namely, the logical structure of non-classificatory concepts is described by means of the same formal tools as in the case of classificatory concepts, that is, tools crucially involving the notion of (ordinary) set. As a result, non-classificatory concepts are reduced to a certain category of classificatory concepts, thereby losing, in fact, their characteristic traits. They are eliminated from the language in favour of expressions whose extensions are ordinary sets. For instance, the use of expressions such as *. . . is intelligent* is restricted, and they are replaced with expressions such as *. . . is more intelligent than. . .* Furthermore, the word *intelligent*, if permitted at all, is only allowed as a typological concept, that is a concept which by definition is a classificatory notion (Pawłowski 1977: 118—124).

Methodologists of humanities realize the need for further development of formal theories that could be applied to these disciplines. The issue of a formally sound and materially adequate description of the language of humanities is a precondition for applying to them, on a large scale, modern-day tools provided by computer science, which is nowadays a sort of historical necessity. In this context, Tadeusz Pawłowski wrote:

> Personally, I pin my hopes on the popular, new disciplines of mathematical-logical type, which deal with any sets of objects, phenomena, or correlations that cannot be defined in a sharp way (fuzzy set theory, fuzzy logic); these disciplines may be efficiently applied in humanities. (Pawłowski 1977: 6)

The idea of a fuzzy set, which was originally conceived in the context of the theory of scientific information, proved fertile in numerous mathematical and methodological disciplines (Negoita, Ralescu 1975: 9—11). Theorists of fuzzy sets have not yet addressed the discussions of the methodologists of humanities concerning the structure of non-classificatory concepts. The above conjecture regarding methodological efficiency of the notion of fuzzy set and its theory encourages one to consider this new approach with an eye to its applicability in the methodology of humanities.

We will focus on the issue of whether it is possible to explicate the meaning of the word "set" formally — i.e. in the framework of some formal theory — in such a way as to ensure that extensions of names — all of them or at least those belonging to a certain class — are sets in the newly defined sense (in addition to extensions of classificatory names); also, the specification of these names in extensional terms should not lead to their reduction to classificatory names. This course of action is contrary to the usual one. For one usually proceeds in such a way as to characterize — as adequately as possible — the logical structure of a non-classificatory name by means of the notion of ordinary set. By contrast, we seek to identify a formally specified meaning of the word "set" such that the extension of a non-classificatory name is a set in this formally specified sense.

Of course, even here we may need procedures regulating the meaning of a name whose extension we consider. The point is, however, that the semantic modification of a given name should not be relevant from the viewpoint of a given language. In particular, no non-classificatory name should be reduced — via formal specification — to a classificatory one.

A formal theory of sets which are extensions of the concepts of a language is the basis of the formal logic of this language. It is possible, therefore, to design a system of logic based on the description of a language in terms of fuzzy sets.

In this article, we will characterize the core of the notion of fuzzy set and discuss the limits of its application. The notion is occasionally misconstrued: fuzzy concepts are wrongly identified with vague concepts and with one-dimensional comparative concepts. We will show that, under certain conditions, extensions of one-dimensional comparative concepts are fuzzy sets. Thus the latter might be called one-dimensional comparative sets. Since all names whose extensions are fuzzy sets — without any qualifications — are one-dimensional comparative names, we suggest that this notion of a set, introduced by Lotfi A. Zadeh, should be labelled a one-dimensional fuzzy set. By zeroing in on propositional logic, we will draw attention to the logic of a language whose expressions have as their extensions one-dimensional fuzzy sets.

Next, we will discuss multidimensional (more-than-one-dimensional) comparative concepts. Extensions of one-dimensional comparative concepts can be regarded — given certain limitations — as fuzzy sets. In order to achieve the same goal in the case of multidimensional comparative concepts, we must further generalize the notion of set. Just as classificatory sets are special cases of one-dimensional fuzzy sets, so one-dimensional fuzzy sets should be special cases of sets in the new sense. The proposed method of generalizing the notion of set results in a broader class of formally characterized concepts of set. Such sets will be called 2-dimensional, 3-dimensional, and — generally — $n$dimensional fuzzy sets. It is also possible to generalize the notion to obtain the concept of infinitely multidimensional set. We will show that — given certain qualifications — extensions of n-dimensional comparative concepts are n-dimensional fuzzy sets. Like in the case of one-dimensional fuzzy sets, we will set out the logic of a language with multidimensional fuzzy concepts.

The discussion of extensions is limited to extensions of nominal expressions. The results, however, can be easily extended to all expressions whose extensions are not sets in the ordinary sense. For example, we can speak of relations whose designata — ordered n-tuples — are elements of their extensions to different degrees. Such relations could be called one-dimensional fuzzy relations or, generally, n-dimensional fuzzy relations.[2]

I. The logical structure of one-dimensional comparative concepts

Given a set of individuals $\mathscr{J}$, we can construct various set-theoretic objects. They include subsets of $\mathscr{J}$, relations, that is subsets of the Cartesian product of $\mathscr{J}$ and itself. Each set of such objects, alone or together with

[2]The extension of a relation $\mathscr{R}$ is a set of all and only those ordered n-tuples of objects — arguments of this relation — such that we can truly say about these objects that they stand in the relation $\mathscr{R}$. Relations whose designata belong to the extension to different degrees should be distinguished from relations whose arguments are elements of extensions of comparative concepts. The example of the former type of relation is the relation . . . *likes*. . . , defined on the set of humans. Arguments of this relation belong to a classificatory set (the set of humans), and the ordered pairs $\langle a, b\rangle$ are elements of the extension of this relation, just as designata of comparative concepts are elements of extensions of these concepts. An example of the second type of relation is the relation . . . *is more visible than*. . . , defined on the set of coloured objects. Two coloured objects, e.g. one red and one blue, are elements of extensions of comparative names. In the theory of measurement, relations of the first type are replaced with relations of the second type, that is, with classificatory relations, whose arguments may fail to be classificatory concepts (Pfanzagl 1971).

sets of other, already constructed, objects, can serve as a starting-point for further constructions. There is an infinite number of such constructible objects, and they can be ordered in a hierarchy. The best-known hierarchy is the hierarchy of types, discovered by Russell and Whitehead. In fact, all other hierarchies draw on their theory.

We will say that set-theoretical objects are of the same type if and only if they were obtained by applying, in the same way, the same construction methods permitted in a given situation (or they *can* be obtained in this way).

Let $\mathscr{J}$ be a set of unconstructed objects (individuals), constituting the domain of a discipline whose language we are considering. Let $\mathscr{R}_{\mathscr{J}}$ denote the family of all sets such that elements of each of them are all and only objects of one type, constructible from the set $\mathscr{J}$ (the hierarchy is 'typically' unambiguous). These are sets of objects which are considered, or could be considered, in a given discipline.[3] We assume that, in a given discipline, the notion of type (in the hierarchy of objects considered in this discipline) is such that one may speak of distinct types of objects provided that these objects are different set-theoretic constructs — for instance, we can say that elements of sets $\mathscr{U}$ and $\mathscr{U} \times \mathscr{U}$ (the Cartesian product of $\mathscr{U}$ and itself) are objects of separate types.

In talking about linguistic expressions, we always bear in mind that they are expressions of one definite language. Besides, we assume that all its expressions have precisely one meaning (although it need not be a classificatory concept). A language which we have in mind does not contain ambiguous expressions. This stipulation allows us, inter alia, to use interchangeably — where it does not lead to misunderstanding — terms "concept" (the meaning of a name) and "name." Yet we do not presuppose that different expressions (expressions of different shapes) are assigned different meanings, that is to say, we do not assume that the language is devoid of synonymous expressions.

A one-dimensional comparative concept "$\mathscr{N}$" — in the most general terms — is a concept such that — given that its designata are elements of the set $\mathscr{T}$, where $\mathscr{T}$ belongs to the family $\mathscr{R}_{\mathscr{J}}$—for each object from $\mathscr{T}$, we can tell — perhaps after a minor regulatory procedure — which object

[3]Strictly speaking, in a hierarchy of objects which are the subject matter of a given discipline, one may distinguish more than one type of individual (Wójcicki 1974: 81—91). The assumption that there are several types of non-constructible objects has no bearing on our discussion. Hence, to simplify matters, we speak of one type.

from $\mathscr{T}$ is $\mathscr{N}$ to a greater, lesser, or equal degree.[4]

Note that in the situation in which we allow for gradation of $\mathscr{N}$, the notion of not-being $\mathscr{N}$ is redundant. Not-being $\mathscr{N}$ is introduced as being $\mathscr{N}$ to a degree lesser than a certain threshold.

Classificatory concepts are special cases of one-dimensional comparative concepts. Namely, they are concepts for whom gradation of being $\mathscr{N}$ is limited to two extreme values. In one case we simply assert that something is an $\mathscr{N}$, while in the other — that something is not an $\mathscr{N}$.

As an example of a one-dimensional comparative concept, consider *tall*. In the ordinary language — i.e. according to the common usage of the word *tall* — we do not divide human beings (in the sense of logical division) into tall and not tall. Rather, it is the case that someone is taller than someone else, or that someone is not taller than someone else, or that someone is as tall as someone else. Two arbitrary persons can be compared with respect to their height.

By "formal structure of a concept" we mean its description in terms of a formal theory, especially logic. Such a description can be provided by specifying the extension of the concept. If two concepts have the same logical structure, that is, if they cannot be distinguished by means of any formal description, then they are mutually substitutable in any contexts without any change in logical properties of those contexts. This is the content of the principle of extensionality. It is the reason why the description of the logical structure should be as complete as possible. The point is that intuitively distinct concepts should possess different logical specifications (different descriptions of logical structures). Otherwise, applying logical tools without restrictions would result in contradictions and paradoxes.

Naturally, one could always give up a language which cannot be logically characterized in favour of a different language, or one could renounce employing formal tools or restrict their application. Dismissing a language, as already mentioned in the introduction, is sometimes impossible, namely, when it can only be replaced by a poorer language which is incapable of fulfilling appropriate cognitive functions. Usually, one would renounce the unrestricted use of formal tools. This solution, however, is inconvenient

[4]It is assumed that the designata of a concept must be objects of the same type. Zadeh stipulates that the designata of a value of a linguistic variable are elements of a universal set (universe of discourse). If values of a linguistic variable X are n-ary relations, whose arguments are elements of universes of discourse $\mathscr{U}_1, \ldots, \mathscr{U}_n$, then the universe of discourse corresponding to the variable X is the Cartesian product of sets $\mathscr{U}_1, \ldots, \mathscr{U}_n$. See Zadeh 1975—1976 (I): 210. This is in accordance with our assumption regarding this case.

if we intend to apply modern-day computer-science apparatus. Thus one should work out formal theories (logic, set theory) whose formal tools enable appropriate specification of the logical structure of a given language.

In Hempel's well-known proposal, the logical structure of a comparative concept is described in terms of two propositional functions of the form $x\mathscr{W}y$ (x precedes y in a relevant respect) and $x\mathscr{R}y$ (x is the same as y in this respect, or — in Hempel's terms — x 'coincides' with y).

Given the relations $\mathscr{W}$ and $\mathscr{R}$ defined on the set $\mathscr{T}$ (where $\mathscr{T}$, like above, is an element of the family $\mathscr{R}_{\mathscr{J}}$, i.e. it is a set of all objects of the same type, constructible from $\mathscr{J}$), we assume that, for any x, y, $z \in \mathscr{T}$ (Hempel 1952: 59; cf. Pawłowski 1977: 109—110):

1.	$x\mathscr{R}x$	(reflexivity of $\mathscr{R}$)
2.	if $x\mathscr{R}y$, then $y\mathscr{R}x$	(symmetry of $\mathscr{R}$)
3.	if $x\mathscr{R}y$ and $y\mathscr{R}z$, then $x\mathscr{R}z$	(transitivity of $\mathscr{R}$)
	— so $\mathscr{R}$ is an equivalence relation —	
4.	if $x\mathscr{R}y$, then it is not the case that $y\mathscr{W}z$	($\mathscr{R}$-irreflexivity of $\mathscr{W}$)
5.	if $x\mathscr{W}y$ and $y\mathscr{W}z$, then $x\mathscr{W}z$	(transitivity of $\mathscr{W}$)
6.	if it is not the case that $x\mathscr{R}y$, then $x\mathscr{W}y$ or $y\mathscr{W}z$	($\mathscr{R}$-connectedness of $\mathscr{W}$).

It is easy to see that the relation $\mathscr{W}^*$, defined as follows:

$x\mathscr{W}^*y$ if and only if $x\mathscr{R}y$ or $x\mathscr{W}y$,

is a partial order, i.e. it is reflexive, transitive, and connected in $\mathscr{T}$.

A relation $\mathscr{R}'$ defined as a relation that holds only between objects such that:

$x\mathscr{W}^*y$ and $y\mathscr{W}^*x$

has the same extension as the relation $\mathscr{R}$ and is the maximal congruence in the relational system $<\mathscr{T}, \mathscr{W}^*, \mathscr{R}'>$.

We will say that $\mathscr{R}'$ is *linked* to$\mathscr{W}^*$ (Wójcicki 1974: 200).

It can be shown that the relation $\mathscr{W}$', defined as follows:

$x\mathscr{W}'y$ if and only if $x\mathscr{W}^*y$ and not $x\mathscr{R}'y$,

is co-extensional with $\mathscr{W}$. We will say that $\mathscr{W}$' is the relation of $\mathscr{R}$-abstraction with respect to $\mathscr{W}^*$.

Such a specification of one-dimensional comparative concepts has two disadvantages (absent from the corresponding specification in terms of fuzzy sets). First, it does not differentiate between, on the one hand, the logical structure of one-dimensional comparative concepts and, on the other, the relation of partial order defined on the designata of these concepts. For instance, on this account, the concept *tall* and the relation of being taller have the same structure. Accordingly, the presence of one-dimensional comparative concepts in a language is treated as merely apparent, or is eliminated. It is replaced by a relation, whose extension is an ordinary set. The second deficiency of this account is that it assigns the same relation to concepts such as *hot*, *warm*, *cold*, *icy*, so that they cease to be logically distinguishable by means of the conceptual apparatus offered by Hempel.

Let us note that a description of the logical structure of comparative concepts in terms of a family of (ordinary) sets is free of these drawbacks. We will not elaborate on this idea here — in fact, it is related to the description involving the notion of fuzzy set. Still, the specification of the logical structure of a concept by means of the notion of fuzzy set has an advantage over the description in terms of family of sets: we define set-theoretic operations on fuzzy sets, whereas there are no such operations for families of sets as arguments of these operations. Furthermore, introducing the notion of fuzzy set to the account of the logical structure of concepts enables a natural generalization of meanings of the terms employed in the formal description of classificatory concepts. In the case of the description of a concept in terms of a family of sets, logical characterization of this concept is provided by means of a set of objects of a type different from the type of the designata of this concept, namely, by means of a family of *sets* of objects of the same type as the designata of the concept. By contrast, in the case of the description in terms of fuzzy sets such a specification will appeal to the (fuzzy) set of designata of the concept.

II. Fuzzy sets

In presenting the notion of fuzzy sets Zadeh points out that extensions of various concepts are vague, and so extensions of these concepts are not sets in the ordinary sense. Like Cantor, in constructing a formal theory of sets, he also draws on the understanding of sets as extensions of names. The idea of the fuzzy set stems from the account of extensions of one-dimensional comparative concepts.

Let us examine Zadeh's method of generalizing the notion of set. The ordinary notion of set (as an extension of a classificatory concept) will be a special case of the notion of set in the generalised sense, so that the ordinary term ”set” will be extensionally subordinate to the term ”set” in the generalised sense.[5]

Let X be a universal set (universe of discourse),and $\mathscr{P}(X)$ — a family of all subsets of X, namely:

$\mathscr{P}(X) = \{A \mid A \subseteq X\}$.[6]

The characteristic function of the set A is the function χ_A defined on X as follows:

$$\chi_A = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if it is not the case that } x \in A \end{cases}$$

Thus the characteristic function χ_A is a mapping of the set X onto the set $\{0,1\}$, namely:

$\chi_A \colon X \rightarrow \{0,1\}$.[7]

Note that there is a mutual one-to-one correspondence between a set and its characteristic function. Due to this correspondence, the discussion of properties of sets can be replaced with the discussion of their characteristic functions.

Assume that:

$Ch(X) = \{ \chi_A \mid A \subseteq X\}$.

Then it is clear that:

$Ch(X) = \{ \chi \mid \chi \colon X \rightarrow \{0,1\}\}$,

[5] The construction of a fuzzy set presented here draws on (Negoita, Ralescu 1975: 12—14).

[6] The expression ”$\{x \mid \alpha(x)\}$” denotes the set of all and only objects which satisfy the function α. The expression ”$A \subseteq B$” means that each element of A is an element of B.

[7] The expression ”$f(x)\colon A \rightarrow B$” means that the function f maps the set A onto the set B. The expression ”$\{a, b, c, \ldots\}$” denotes the set whose only elements are a, b, c, ...

that is to say, the set $Ch(X)$ is identical with the set of all mappings of the set X onto the set $\{0,1\}$.

We can now define operations $\vee$, $\wedge$, $\neg$ in the set of characteristic functions $Ch(X)$.

$(\chi \vee \chi')(x) = \max[\chi(x), \chi'(x)]$, that is, the characteristic function $\chi \vee \chi'$, which is the product of performing the operation "$\vee$" on characteristic functions χ and χ', assumes — for the argument x — the greater out of two values, $\chi(x)$ and $\chi'(x)$, i.e. $(\chi \vee \chi')(x)$ assumes value 1 if and only if at least one of the values $\chi(x)$, $\chi'(x)$ has 1 as its value.

$(\chi \wedge \chi')(x) = \min[\chi(x), \chi'(x)]$, that is, the characteristic function $\chi \wedge \chi'$, which is the product of performing the operation "$\wedge$" on characteristic functions χ and χ', assumes — for the argument x — the lesser out of two values, $\chi(x)$ and $\chi'(x)$, i.e. $(\chi \wedge \chi')(x)$ assumes value 1 if and only if both $\chi(x)$ and $\chi'(x)$ have 1 as their value.

$\neg\chi(x) = 1 - \chi(x)$, i.e. the characteristic function $\neg\chi(x)$, which is the product of performing the operation "$\neg$" on χ, assumes — for the argument x — a value which is equal to the arithmetic result of subtracting $\chi(x)$ from 1. Thus $\neg\chi(x)$ assumes value 1 if and only if $\chi(x)$ has value 0.

It is easy to show that algebras $(\mathscr{P}(X), \cup, \cap, ')$ and $(Ch(X), \vee, \wedge, \neg)$ are isomorphic.

Let us construct a homomorphic extension of the algebra $(Ch(X), \vee, \wedge, \neg)$ in such a way as to replace the set of characteristic functions $Ch(X)$ with the set of all functions from the set X to the set [0,1].[8] These functions will be called characteristic functions of fuzzy sets. We will distinguish characteristic functions of fuzzy sets from (ordinary) characteristic functions by attaching the symbol "~".

$$\widetilde{Ch}(X) = \{ \tilde{\chi} \mid \tilde{\chi}: X \to [0,1]\},$$

The homomorphism of both algebras of characteristic functions preserves the operations. In the algebra of characteristic functions of fuzzy sets, these operations are denoted by the same symbols that were used in the algebra of (ordinary) characteristic functions. They are defined as follows:

$$(\tilde{\chi} \vee \tilde{\chi}')(x) = \max[\tilde{\chi}(x), \tilde{\chi}'(x)]$$
$$(\tilde{\chi} \wedge \tilde{\chi}')(x) = \min[\tilde{\chi}(x), \tilde{\chi}'(x)]$$
$$\neg \tilde{\chi}(x) = 1 - \tilde{\chi}(x).$$

[8] "[0,1]" denotes the set of all real numbers x, $0 \leqslant x \leqslant 1$.

The algebra ($\mathscr{P}(X)$, $\cup$, $\cap$, ') is isomorphic to the algebra ($Ch(X)$,$\vee$, $\wedge$, $\neg$), so we can say that a set (in the ordinary sense) is a characteristic function. Similarly, we can say that a fuzzy set is a function from the set X to the set [0,1], i.e. an element of the set $\widetilde{Ch}(X)$. The algebra of fuzzy sets, therefore, is a homomorphic extension of the algebra of (ordinary) sets.

Let (X) be the set of all fuzzy sets which can be constructed from the elements of the universal set X. To distinguish fuzzy sets from (ordinary) sets, we will use the symbol "~". Where a symbol of a fuzzy set is an index of a symbol of a characteristic function of a fuzzy set, the sign "~" will be used only once, over a symbol of a characteristic function, not the fuzzy set.

Based on these terminological decisions:

$$\widetilde{P}(X) = \{ \widetilde{A} |\ \widetilde{\chi}_A : X \rightarrow\ [0{,}1]\}.$$

The set $\widetilde{A}$ is a subset of $\widetilde{B}$ (symbolically, $\widetilde{A} \subseteq\ \widetilde{B}$) if and only if $\forall x[\ \widetilde{\chi}_A(x) \leqslant\ \widetilde{\chi}_B(x)]$.

Two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ are identical if $\forall x[\ \widetilde{\chi}_A(x) = \widetilde{\chi}_B(x)]$, i.e.:

$$\widetilde{A} = \widetilde{B} \text{ if and only if } \widetilde{\chi}_A = \widetilde{\chi}_B.$$

It is easy to see that $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A} \subseteq\ \widetilde{B}$ and $\widetilde{B} \subseteq\ \widetilde{A}$.

In what follows, if the context unequivocally determines which sense is relevant, we will use the word "set" instead of the term "fuzzy set."

Operations on fuzzy sets will be denoted by the same symbols as in the case of operations on ordinary sets.

A set $\widetilde{C}$ is a union of $\widetilde{A}$ and $\widetilde{B}$ if and only if:

$$\widetilde{\chi}_C(x) = \max[\ \widetilde{\chi}_A(x),\ \widetilde{\chi}_B(x)]$$

That is to say:

$$\widetilde{C} = \widetilde{A} \cup \widetilde{B} \text{ if and only if } \widetilde{\chi}_C = \widetilde{\chi}_A \vee \widetilde{\chi}_B.$$

A set $\widetilde{C}$ is an intersection of $\widetilde{A}$ and $\widetilde{B}$ if and only if:

$$\widetilde{\chi}_C(x) = \min[\ \widetilde{\chi}_A(x),\ \widetilde{\chi}_B(x)]$$

That is to say:

$\widetilde{C} = \widetilde{A} \cap \widetilde{B}$ if and only if $\widetilde{\chi}_C = \widetilde{\chi}_A \wedge \widetilde{\chi}_B$.

A set $\widetilde{B}$ is a complement of $\widetilde{A}$ if and only if:

$\widetilde{\chi}_B(x) = 1$ - $\widetilde{\chi}_A(x)$.

That is to say:

$\widetilde{B} = \widetilde{A}'$ if and only if $\widetilde{\chi}_B = \neg\widetilde{\chi}_A$

Notice that there is a function which assumes value 1 alone and a function that always assumes value 0, namely, for each x: $\widetilde{\chi}_x(x) = 1$; $\widetilde{\chi}_\emptyset(x) = 0$.

We will show that the algebra $(\widetilde{Ch}(X), \vee, \wedge, \neg)$ is quasi-Boolean (De Morgan algebra). From this it will follow that the algebra of fuzzy sets $(\widetilde{P}(X), \cup, \cap, ')$ — which is isomorphic to the former — is also quasi-Boolean.

An abstract algebra $(A, \cup, \cap)$ is called *lattice* if for all a, b, c $\in A$, the following conditions are met:

1.	$a \cup b = b \cup a$	$a \cap b = b \cap a$
2.	$a \cup (b \cup c) = (a \cup b) \cup c$	$a \cap (b \cap c) = (a \cap b) \cap c$
3.	$(a \cap b) \cup b = b$	$a \cap (a \cup b) = a$.

A lattice is called *distributive* just in case it satisfies the additional conditions:

4.	$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$	$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

An abstract algebra $(A, \cup, \cap)$ is quasi-Boolean when $(A, \cup, \cap)$ is a distributive lattice with an individual element V and a unary operation ' defined on A, and the following conditions are met:

5. $a = a''$
6. $(a \cup b)' = a' \cap b'$

The algebra of characteristic functions of fuzzy sets fulfils all requirements 1—6. We will show this fact only in the last three points.

ad 4. (distributivity)

$[(\tilde{\chi}_1 \wedge (\tilde{\chi}_2 \vee \tilde{\chi}_3)](x) = \min[\tilde{\chi}_1(x), (\tilde{\chi}_2 \vee \tilde{\chi}_3)(x)] = \min[\tilde{\chi}_1(x), \max(\tilde{\chi}_2(x), \tilde{\chi}_3(x))]$.Thus $\min[\tilde{\chi}_1(x), (\tilde{\chi}_2 \vee \tilde{\chi}_3)(x)] =$ either $\min[\tilde{\chi}_1(x), (\tilde{\chi}_2(x)]$ or $\min[\tilde{\chi}_1(x), (\tilde{\chi}_3(x)]$. Hence $\min[\tilde{\chi}_1(x), (\tilde{\chi}_2 \vee \tilde{\chi}_3)(x)] = \max[\min(\tilde{\chi}_1(x), (\tilde{\chi}_2(x)), \min(\tilde{\chi}_1(x), (\tilde{\chi}_3(x))]$. So eventually, for each x, $[(\tilde{\chi}_1 \wedge (\tilde{\chi}_2 \vee \tilde{\chi}_3)](x) = (\tilde{\chi}_1 \wedge \tilde{\chi}_2) \vee (\tilde{\chi}_1 \wedge \tilde{\chi}_3)$.

ad 5. $\neg\tilde{\chi}(x) = 1 - (1 - \tilde{\chi}(x)) = \tilde{\chi}(x)$.

ad 6. $\neg(\tilde{\chi}_1 \vee \tilde{\chi}_2) = 1 - (\tilde{\chi}_1 \vee \tilde{\chi}_2)(x) = 1 - \max[(\tilde{\chi}_1(x), \tilde{\chi}_2(x)] = \min[1 - \tilde{\chi}_1(x), 1 - \tilde{\chi}_2(x)] = \min[\neg\tilde{\chi}_1(x), \neg\tilde{\chi}_2(x)] = \neg\tilde{\chi}_1(x) \wedge \neg\tilde{\chi}_2(x)$.

Note that the following expressions are not laws of the algebra $\widetilde{Ch}(X)$,$\vee$, $\wedge$, $\neg$):

$\neg(\tilde{\chi} \wedge \neg\tilde{\chi}) = V; \neg\tilde{\chi} \vee \tilde{\chi} = V$

When $\tilde{\chi}(a) = \frac{1}{2}$, then $\neg\ \tilde{\chi}(a) = \frac{1}{2}$, so: $[\neg(\tilde{\chi} \wedge \neg\tilde{\chi})](a) = \frac{1}{2}$ and $(\neg\tilde{\chi} \vee \tilde{\chi})(a) = \frac{1}{2}$.

III. Extensions of one-dimensional comparative concepts as fuzzy sets

Identifying the extension of a concept consists in defining the set of designata (referents) of this concept. As we said, extensions of one-dimensional comparative concepts are not sets in the ordinary sense. In the case of one-dimensional comparative concepts designata can be compared with each other with respect to the degrees of membership in the set. In the case of fuzzy sets, we can assign to each object a value from the interval [0,1] with which the object is an element of a given set. A one-dimensional concept $\mathscr{N}$ of the logical structure $(\mathscr{T}, \mathscr{W}, \mathscr{R})$ can be logically characterized by means of a fuzzy set if and only if the relational system $(\mathscr{T}, \mathscr{W}^*)$ can be embedded in the system ([0,1], $\leqslant$).

The issue of embedding one system in another is a well-known problem — for instance, in the theory of measurement.

If the cardinality of $\mathscr{W}$ is greater than that of [0,1], then it is impossible to embed $(\mathscr{T}, \mathscr{W}^*)$ in ([0,1], $\leqslant$).[9] Thus a logical specification of one-dimensional comparative concepts by means of fuzzy sets is not always possible. In the case of one-dimensional comparative concepts which can be characterized by means of fuzzy sets, we are in a position to declare that we

[9]Zadeh (1965) is aware of the possibility of using a different set instead of the interval [0,1].

have provided a description of their logical structure. To simplify matters, we can say that extensions of such concepts are fuzzy sets.

A logical characterization of one-dimensional comparative concepts by means of fuzzy sets does not result in the reduction of these concepts to classificatory concepts. It also allows room for a different analysis of concepts which are intuitively different despite the fact that they share the description of logical structure of the type proposed by Hempel — e.g. *tepid*, *warm*, *hot*. Lukewarm, warm, and hot objects will be elements of corresponding fuzzy sets (by means of which we will specify the concepts *tepid*, *warm*, *hot*) to different degrees.

This example makes it clear that the specification of the words *tepid*, *warm*, *hot* in terms of fuzzy sets can be correctly carried out by means of methods of the theory of measurement. One could even say that, in the light of such an account, the theory of measurement becomes the method of a correct implementation of logical specification. Thus the theory of measurement proves to be an integral part of metamathematics, broadly understood (Wójcicki 1974: 10).

In the family of fuzzy sets, we define operations "$\subseteq$ " and "=", which are generalizations of corresponding relations in the class of classificatory sets. Consequently, it is possible to describe relations holding between extensions of one-dimensional comparative concepts (given that these concepts are specified by means of fuzzy sets). Of course, the terminology used for such a description can also be generalized in a natural way.

We will say that a logical characterization of a language is adequate only if all concepts of that language which intuitively have distinct extensions receive different logical specifications.

Suppose that a given language can be adequately characterized by means of fuzzy sets. Then the logic of this language can be formalized. Here, we will limit ourselves to propositional logic.

Clearly, we must admit to more than one truth-value. For instance, truth-values
vary depending on the degree of membership of the objects in the set. There is sufficient reason to assume that it is convenient to equate the set of truth-values with the set [0,1], i.e. the set whose elements serve to mark the degrees of membership of an object in an extension of name. Of course, it leaves open the issue of distinguished truth-values.

A connective is truth-functional if the truth-value of the compound sentence constructed by means of this connective is determined by the truth-values of sentence-arguments of this connective. Among the definable truth-functional connectives, only those are of concern here which are extensions of ordinary connectives of two-valued logic, namely, negation, disjunction, conjunction (of course we are also interested in implication, yet we assume its definability in terms of the above connectives). To define these connectives, it is enough to associate them with calculations in the algebra of characteristic functions of fuzzy sets. We calculate truth-values of sentences — depending on the truth-values of sentence-arguments — in the same way as we calculate degrees of membership in fuzzy sets.

In the ordinary way of speaking, we do not assess the truth-value of sentences by pointing to a number from the interval [0,1]. Such an assessment is carried out by means of a certain vocabulary, dependent on a given language. But notice that in the case of a language composed of one-dimensional comparative concepts, the expressions describing truth-values of sentences of that language are also one-dimensional comparative concepts. Thus membership of any sentences of that language in the set of sentences with a specific truth-value denoted in this language (or, more precisely, in its metalanguage) is mutually comparable. Suppose that $\mathfrak{J}$ is a word denoting a truth-value. For any two sentences, we can tell whether they belong to the set of sentences with the truth-value $\mathfrak{J}$ to an equal degree or to different degrees.

In any case, however, the set of expressions used to describe truth-values should contain two expressions — *true*, *false* (or their equivalents). Such a set can be extended, for instance, by adding expressions such as: *very true*, *fairly true*, *not very true*, *not false* (as for fuzzy logic, cf. Zadeh 1975—1976, esp. part II, section 3). We will not go into details of this broad issue. Let us notice, however, that if we regard 1 alone as the distinguished value, then the algebra of truth-values ([0,1], 1, $\vee$, $\wedge$, $\neg$) will be quasi-Boolean. This means, in particular, that expressions such as $\alpha \vee \neg\, \alpha$, $\neg\, (\alpha \wedge \neg\, \alpha)$ will not be tautologies. Which, in fact, is in accordance with intuitions. Consider the word *tall*. The sentences *John is tall* and *John is not tall* are not logically contradictory. After all, it is not the case that one of them must be true and the other false. The sentence *John is taller John is not tall* is true if one of the disjuncts is true, yet none of these sentences have to be true. The sentence *John is tall and John is not tall* is false when one of the conjuncts is false, but none of these sentences have to be false.

IV. Multidimensional comparative concepts

Under certain conditions, fuzzy sets can be employed in an extensional specification of one-dimensional comparative concepts. Languages composed of such concepts can be adequately described in terms of the theory of fuzzy sets. It is easy to see, however, that fuzzy sets are not sufficient for a satisfactory description of languages which contain comparative concepts of more than one dimension. They are not suited for a satisfactory description in the same sense in which they were suited for a description of one-dimensional comparative concepts. In particular, an extensional description of multidimensional comparative concepts leads to their reduction to one-dimensional comparative concepts. A question arises, therefore, whether it is possible to generalize the notion of a fuzzy set so as to make extensions of multidimensional comparative concepts fuzzy (perhaps under conditions analogous to the requirements accepted in the case of one-dimensional comparative concepts).

Loosely speaking, an n-dimensional comparative concept is a concept such that all its designata are simultaneously designata of n different one-dimensional comparative concepts. For example, shirt size is a two-dimensional comparative concept. It consists of the collar size and the chest size.

The above definition of n-dimensional comparative concept reveals a crucial difference between, on the one hand, the division into classificatory and comparative concepts, and, on the other, the division into one-dimensional and multidimensional comparative concepts. Classificatory concepts are one-dimensional comparative concepts, but no n-dimensional comparative concept is an $(n+1)$-dimensional comparative concept.

n-dimensional comparative concepts (according to Hempel) have the following logical structure, involving a system of pairs of relations:

$$(\mathscr{W}_i, \mathscr{R}_j),$$

where $i = 1, 2, \ldots, n$.

For each pair, we have the same requirements as those imposed on relations $\mathscr{W}$ and $\mathscr{R}$ with regard to specification of one-dimensional comparative concepts. Also, the field of each relation $\mathscr{W}_i$ is a set $\mathscr{T}$, i.e. a set from the family $\mathscr{R}_{\mathscr{J}}$ such that the designata of a given concept are elements of that set.

In the case of one-dimensional comparative concepts, we have proposed a generalization of the notion of set by adding values to the set of values of the characteristic function of an (ordinary) classificatory set. Thus a classificatory set is a fuzzy set whose characteristic function accepts only 0 and 1 as values. However, if we were to invoke fuzzy sets in order to specify extensions of multidimensional comparative concepts, we would not be able to offer a solution according to which a set in a more general sense is also a fuzzy set. It follows from the above-discussed fact that a classificatory concept is a one-dimensional comparative concept, but an n-dimensional comparative concept ($n > 1$) is not a one-dimensional comparative concept but a *class* of one-dimensional comparative concepts. Thus when we talk about a further generalization of the notion of set and say that a fuzzy (and classificatory) set is a special case of such a set, then we speak of a generalization in a different sense than in the case of a generalization of the notion of classificatory set to the notion of fuzzy set. It will be a kind of generalization analogous to the generality of the notion of n-dimensional ($n > 1$) comparative *concept* with respect to the notion of one-dimensional comparative concept.

The phrase "n-dimensional fuzzy set" will refer to the following construction.

Let ${}^{n}\chi$ be an arbitrary function from the set X to the set $[0,1]^n$.[10] That is:

${}^{n}\chi\colon X \rightarrow [0,1]^n$.

Let $Ch_n(X)$ be the set of all functions ${}^{n}\chi$, that is:

$Ch_n = \{{}^{n}\chi \mid {}^{n}\chi\colon X \rightarrow [0,1]^n\}$.

For $n = 1$, $Ch_n(X) = (\widetilde{Ch}X)$, that is to say, the set $Ch_1(X)$ is identical to the set of characteristic functions of fuzzy sets.

Any function ${}^{n}\chi \in Ch_n$ can be presented as an n-tuple of functions:

${}^{n}\chi = < \widetilde{Ch}^1, \widetilde{Ch}^2, \ldots, \widetilde{Ch}^n >$,

where $\widetilde{Ch}^1 \in \widetilde{Ch}(X)$.

Consider the algebra $\widetilde{Ch}_n(X), \vee, \wedge, \neg)$ in which the operations $\vee$, $\wedge$, $\neg$ are defined as follows:

[10] A^n is an n-times Cartesian product of the set A and itself, that is, it is a set of all ordered n-tuples of elements of A.

$$^n\chi_A \vee {}^n\chi_B = <\tilde{\chi}_A^1 \vee \tilde{\chi}_B^1, \ldots, \tilde{\chi}_A^n \vee \tilde{\chi}_B^n>,$$

where $(\tilde{\chi}_A^i \vee \tilde{\chi}_B^i)(x) = \max[\tilde{\chi}_A^i(x), \tilde{\chi}_B^i(x)]$;

$$^n\chi_A \wedge {}^n\chi_B = <\tilde{\chi}_A^1 \wedge \tilde{\chi}_B^1, \ldots, \tilde{\chi}_A^n \wedge \tilde{\chi}_B^n>,$$

where $(\tilde{\chi}_A^i \wedge \tilde{\chi}_B^i)(x) = \min[\tilde{\chi}_A^i(x), \tilde{\chi}_B^i(x)]$;

$$\sim {}^n\chi_A = <\sim\tilde{\chi}_A^1, \ldots, \sim\tilde{\chi}_A^n>,$$

where $(\sim\tilde{\chi}_A^i)(x) = 1 - \tilde{\chi}_A^i(x)$.

The algebra $\widetilde{Ch}_n(X)$,∨, ∧, ¬) is a homomorphic extension of the algebra $\widetilde{Ch}(X)$,∨, ∧, ¬)). It is quasi-Boolean.

The algebra of n-dimensional fuzzy sets is the algebra of sets isomorphic to the algebra of functions $Ch_n(X)$, called characteristic functions of n-dimensional fuzzy sets. An n-dimensional fuzzy set is an isomorphic image of an n-dimensional characteristic function.

Are extensions of n-dimensional comparative concepts n-dimensional fuzzy sets?

If an extension of an n-dimensional comparative concept, of a logical structure defined by the system $(\mathscr{T},_1, \ldots, \mathscr{W}_n)$, is to be treated as an n-dimensional fuzzy set, it is necessary, and sufficient, that each relational system $(\mathscr{T},\mathscr{W}_i)$, $1 \leqslant i \leqslant n$, could be embedded in the system ([0,1], $\leqslant$].

It is clear, therefore, that — like in the case of one-dimensional comparative concepts — such an embedment is impossible if, for some i, cardinality of $\mathscr{W}_i$ exceeds cardinality of [0,1]. Thus, in theory, not every n-dimensional comparative concept can be extensionally characterized by means of an n-dimensional fuzzy set. Concepts which can be specified in this way can be labelled n-dimensional fuzzy concepts.

The problem of logic for a language whose concepts are n-dimensional comparative concepts can be solved analogously to the problem of logic for a language composed of one-dimensional comparative concepts. As the set of truth-values, we should take the set $[0,1]^n$. Phrases used to assess truth-values of sentences should be n-dimensional comparative concepts as well. The point is that sentences which assert membership of objects in a given set should differ in truth-value when those objects belong to that set to different degrees.

The above discussion indicates prospects and limitations of the idea of a fuzzy set. In particular, it shows that this notion is in need of further generalization. In this regard, it would be interesting to compare fuzzy sets in a broad sense with the special theory of sets developed by Polish computer scientists. It is worth considering the advantages and disadvantages of both accounts with an eye on the methodological problems associated with concepts in the humanities.

Bibliography

1. Ajdukiewicz, Kazimierz (1974) *Pragmatic Logic*. Dordrecht: Reidel.

2. Hempel, Carl Gustav (1952) *Fundamentals of Concept Formation in Empirical Science*. Chicago (IL): University of Chicago Press.

3. Negoita, Constantin V. and Dan A. Ralescu (1975) *Applications of Fuzzy Sets to Systems Analysis*. Basel: Springer.

4. Pawłowski, Tadeusz (1977) *Pojęcia i metody współczesnej humanistyki*. Wrocław: Zakład Narodowy im. Ossolińskich.

5. Pfanzagl, Johann (1971) *Theory of Measurement*. Berlin: Springer.

6. Wójcicki, Ryszard (1974) *Metodologia formalna nauk empirycznych*. Wrocław: Zakład Narodowy im. Ossolińskich.

7. Zadeh, Lotfi A. (1965) "Fuzzy Sets." *Information and Control* 8: 338—353.

8. Zadeh, Lotfi A. (1975-1976) "The Concept of a Linguistic Variable and its Application to Approximate Reasoning." *Information Sciences* 8 (1975): 199—251 (I), 301—357 (II); *Information Sciences* 9 (1976) 43—80 (III).